

Volume-preserving maps between Hermitian symmetric spaces of compact type

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1 Introduction

Let M be an irreducible Hermitian symmetric space of compact type, equipped with a canonical Kähler-Einstein metric ω . Then ω^n is the associated volume form (up to a positive constant depending only on n). We establish in this paper the following rigidity theorem:

Theorem 1.1. *Let $F = (F_1, \dots, F_m)$ be a holomorphic mapping from a connected open subset $U \subset M$ into the m -Cartesian product $M \times \dots \times M$ of M . Assume that each F_j is generically non-degenerate in the sense that $F_j^*(\omega^n) \not\equiv 0$ over U . Assume that F satisfies the following volume-preserving (or measure-preserving) equation:*

$$\omega^n = \sum_{i=1}^m \lambda_i F_i^*(\omega^n), \quad (1)$$

for certain constants $\lambda_j > 0$. Then for each j with $1 \leq j \leq m$, F_j extends to a holomorphic isometry of (M, ω) . In particular, it holds that $\sum_{j=1}^m \lambda_j = 1$.

Rigidity properties are among the fundamental phenomena in Complex Analysis and Geometry with several variables, that seek for the global extension and uniqueness for various holomorphic objects up to certain group actions. The rigidity problem that we are concerned with in this paper was initiated by a celebrated paper of Calabi [Ca]. In [Ca], Calabi studied the global holomorphic extension and uniqueness (up to the action of the holomorphic isometric group of the target space) for a local holomorphic isometric embedding between two Kähler manifolds. He established the global extension and the Bonnet type rigidity theorem for a local holomorphic isometric embedding from a complex manifold with a real analytic Kähler metric into a standard complex space form. The phenomenon discovered by Calabi [Ca] has been further explored in the past several decades due to its extensive connections with many problems in Analysis and Geometry. (See [U] [DL] [DL1], for instance).

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In 2004, motivated by the modularity problem of the algebraic correspondences in algebraic number theory, Clozel and Ullmo [CU] were led to study the rigidity problems for local holomorphic isometric maps and even more general volume-preserving maps between bounded symmetric domains equipped with their Bergman metrics. More precisely, by reducing the modularity problem to the rigidity problem for local holomorphic isometries, Clozel-Ullmo proved that an algebraic correspondence in the quotient of a bounded symmetric domain preserving the Bergman metric has to be a modular correspondence in the case of the unit disc in the complex plane and in the case of bounded symmetric domains of rank ≥ 2 . Notice that in the one dimensional setting, volume preserving maps are identical to the metric preserving maps. Thus the Clozel-Ullmo result also applies to the volume preserving algebraic correspondences in the lowest dimensional case. Motivated by the work in [CU], Mok carried out a systematic study of the rigidity problem for local isometric embeddings in a very general setting. Mok in [Mo2-4] proved the total geodesy for a local holomorphic isometric embedding between bounded symmetric domains D and Ω when either (i) the rank of each irreducible component of D is at least two or (ii) $D = \mathbb{B}^n$ and $\Omega = (\mathbb{B}^n)^p$ for $n \geq 2$. In a paper of Yuan-Zhang [YZ], the total geodesy is obtained in the case of $D = \mathbb{B}^n$ and $\Omega = \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_p}$ with $n \geq 2$ and N_l arbitrary for $1 \leq l \leq p$. Earlier, Ng in [Ng2] had established a similar result when $p = 2$ and $2 \leq n \leq N_1, N_2 \leq 2n - 1$. In a paper of Yuan and the second author of this paper [HY1], we established the rigidity result for local holomorphic isometric embeddings from a Hermitian symmetric space of compact type into the product of Hermitian symmetric spaces of compact type with even negative conformal factors where certain non-cancellation property for the conformal factors holds. (This cancellation condition turns out to be the necessary and sufficient condition for the rigidity to hold due to the presence of negative conformal factors.) In a recent paper of Ebenfelt [E], a certain classification, as well as its connection with many important problems in CR geometry, has been studied for local isometric maps when the cancellation property fails to hold. The recent paper of Yuan [Y] studied the rigidity problem for local holomorphic maps preserving the (p, p) -forms between Hermitian symmetric spaces of non-compact type.

The work of Clozel and Ullmo has left open an important question of understanding the modularity problem for volume-preserving correspondences in the quotient of Hermitian symmetric spaces of higher dimension equipped with their Bergman metrics. In 2012, Mok and Ng answered, in the affirmative, the question of Clozel and Ullmo in [MN] by establishing the rigidity property for local holomorphic volume preserving maps from an irreducible Hermitian manifold of non-compact type into its Cartesian products.

The present paper continues the above mentioned investigations, especially, those in [CU], [MN] and [HY1]. Our main purpose is to establish the Clozel-Ullmo and Mok-Ng results for local measure preserving maps between Hermitian symmetric spaces of compact type. Notice that in the one complex dimensional Riemann sphere setting, Theorem 1.1 also follows from the isometric rigidity result obtained in an earlier paper of the second author with Yuan [HY1].

We should mention many highly related studies for the rigidity property of holomorphic maps or CR maps. Here, we only quote the papers [B] [DA] [Fu] [JH] [Hu1-2] [HY2] [Ji] [KZ]

[Mo1] [Mo5] [MN1] [Ng] [Ng1] and many references therein to name a few.

We now briefly describe the organization of the paper and the basic ideas for the proof of Theorem 1.1. The major difficulty in proving Theorem 1.1 is to obtain the algebraicity for a certain component F_j from F . For this, we introduce the concept of Segre family for an embedded projective subvariety. Notice that in all the previous work, Segre varieties were only defined for a real submanifold in a complex space through complexification. Our Segre family is defined by slicing the manifold with a hyperplane in the ambient projective space, associated with points in its conjugate space. The Segre family is invariant under holomorphic isometric transformations, whose defining function is closely related to the potential function of the canonical metric. The first step in our proof is to show that a certain component F_j preserves at least locally the Segre family. The next difficult step is then to show that this gives the algebraicity of F_j . The proof is based on a case by case analysis according to the type of the space. After the algebraicity is established, we will further show that F_j extends to a birational self-map of the space and then must be identical to a holomorphic self-isometry of the space. Once this is done, we can delete F_j from the original equation and then apply an induction to conclude the rigidity for other components.

The organization of the paper is as follows: In §2, we first introduce the Segre family for a polarized projective variety. We then describe the canonical and minimal embedding of the space into a complex projective space in terms of the type of the space. In §3, we derive a general theorem for degenerate holomorphic embeddings which will play a fundamental role in the later development. In §4, we provide the algebraicity for one of the components of the holomorphic mapping F under additional assumptions which include the non-degeneracy condition introduced in §3. In §5, we show that the non-degeneracy holds for any Hermitian symmetric space of compact type. The argument in §5-§7 varies as the type of the space varies and thus has to be done case by case. §6 and §7 are devoted to verifying several important properties concerning the Segre families of the space, that are needed in §4. We also include two Appendices for convenience of the reader, in which we prove some folk-lore results that are crucial in our paper.

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2 Irreducible Hermitian symmetric spaces and their Segre varieties

2.1 Segre varieties of projective subvarieties

Write $z = (z_1, \dots, z_n, z_{n+1})$ for the coordinates of \mathbb{C}^{n+1} and $[z] = [z_1, \dots, z_n, z_{n+1}]$ for the homogeneous coordinates of \mathbb{P}^n . For a polynomial $p(z)$, we define $\overline{p}(z) := p(\overline{z})$. For a projective

variety $V \subset \mathbb{P}^n$, write \mathcal{I}_V for the ideal consisting of homogeneous polynomials in z that vanish on V . We define the conjugate variety V^* of V to be the projective variety defined by $\mathcal{I}_V^* := \{\bar{f} : f \in \mathcal{I}_V\}$. Apparently the map $z \mapsto \bar{z}$ defines a diffeomorphism from V to \bar{V} . When \mathcal{I}_V has a basis consisting of polynomials with real coefficients, $V^* = V$. Also if V is irreducible and has a smooth piece parametrized by a neighborhood of the origin of a complex Euclidean space through polynomials with real coefficients, then $V^* = V$.

Next for $[\xi] \in V^*$, we define the Segre variety Q_ξ of V associated with ξ by $Q_\xi = \{[z] \in V : \sum_{j=1}^{n+1} z_j \xi_j = 0\}$ which is a subvariety of codimension one in V . Similarly, for $[z] \in V$, we define the Segre variety Q_z^* of V^* associated with z by $Q_z^* = \{[\xi] \in V^* : \sum_{j=1}^{n+1} z_j \xi_j = 0\}$. It is clear that $[z] \in Q_\xi$ if and only if $[\xi] \in Q_z^*$. The Segre family of V is defined to be the projective variety $\mathcal{M} := \{([z], [\xi]) \in V \times V^*, [z] \in Q_\xi\}$.

Now, we let (M, ω) be an irreducible Hermitian symmetric space of compact type canonically embedded in a certain minimal projective space \mathbb{P}^N , that we will describe in detail later in this section. Then under this embedding, its dual space M^* is just M itself. Taking ω to be the natural restriction of the Fubini-Study metric to M , the holomorphic isometric group of M is then the restriction of a subgroup of the unitary actions of the ambient space. Now, for two points $p_1, p_2 \in M$, let U be an $(N+1) \times (N+1)$ unitary matrix such that $\sigma([z]) = [z] \cdot U$ is an isometry sending p_1 into p_2 . Then $\sigma^*([\xi]) = [\xi] \bar{U}$ is an isometry of M^* . By a straightforward verification, we see that σ^* biholomorphically sends $Q_{p_1}^*$ to $Q_{p_2}^*$. Similarly, for any $q_1, q_2 \in M^*$, Q_{q_1} is unitary equivalent to Q_{q_2} . In the canonical embeddings which we will describe later, the hyperplane section at infinity of the manifold is a Segre variety. Since the one at infinity is built up from Schubert cells and all Segre varieties are holomorphically equivalent to each other, one deduces that each Segre variety of M is irreducible. We will give a detailed proof of this fact without using the Schubert decomposition in §7. This fact will play a role in the proof of our main theorem.

2.2 Canonical embeddings

We now describe a certain canonical embedding of the Hermitian symmetric space M of compact type into \mathbb{P}^N . This embedding will be played a crucial role in our computation.

♣1. Grassmanians (Type I Hermitian symmetric spaces of compact type): Write $G(p, q)$ for the Grassmanian space consisting of p planes in \mathbb{C}^{p+q} . (Since $G(p, q)$ is biholomorphically equivalent to $G(q, p)$, we will assume $p \leq q$ in what follows).

There is a matrix representation of $G(p, q)$ as the equivalence classes of $p \times (p+q)$ non-degenerate matrices under the matrix multiplication from the left by elements of $GL(p, \mathbb{C})$. A Zariski open affine chart \mathcal{A} for $G(p, q)$ is identified with \mathbb{C}^{pq} with coordinates Z for elements of the form:

$$(I_{p \times p} \quad Z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & z_{11} & z_{12} & \cdots & z_{1q} \\ 0 & 1 & 0 & \cdots & 0 & z_{21} & z_{22} & \cdots & z_{2q} \\ & & & \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & z_{p1} & z_{p2} & \cdots & z_{pq} \end{pmatrix}, \text{ where } Z \text{ is a } n \times n \text{ matrix.}$$

The Plücker embedding $G(p, q) \rightarrow \mathbb{P}(\wedge^p \mathbb{C}^{p+q})$ is given by mapping the p -plane Λ spanned by vectors $v_1, \dots, v_p \in \mathbb{C}^{p+q}$ into the wedge product $v_1 \wedge v_2 \wedge \dots \wedge v_p \in \wedge^p \mathbb{C}^{p+q}$. It is invariant under the action of $SU(p+q)$. In homogenous coordinates, the embedding is given by the $p \times p$ minors of the $p \times (p+q)$ matrices (up to a sign). More specifically, in the above local affine chart, we have the following:

$$Z \rightarrow [1, Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}, \dots] \quad (2)$$

which is denoted for simplicity of notation, in what follows, by $[1, r_z] = [1, \psi_1, \psi_2, \dots, \psi_N]$.

Here and in what follows, $Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$ is the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, \dots, i_k^{\text{th}}$ rows and $j_1^{\text{th}}, \dots, j_k^{\text{th}}$ columns, where the indices run through

$$k = 1, 2, \dots, p, 1 \leq i_1 < i_2 < \dots < i_k \leq p, 1 \leq j_1 < j_2 < \dots < j_k \leq q.$$

Notice that when $k = 1$, $Z \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} = z_{i_1 j_1}$.

Notice that under such an embedding into the projective space, $(G(p, q))^* = G(p, q)$. We thus have the same affine coordinates for $(G(p, q))^*$:

$$(I_{p \times p} \quad \Xi) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \xi_{11} & \xi_{12} & \dots & \xi_{1q} \\ 0 & 1 & 0 & \dots & 0 & \xi_{21} & \xi_{22} & \dots & \xi_{2q} \\ & & & \dots & & & & \dots & \\ 0 & 0 & 0 & \dots & 1 & \xi_{p1} & \xi_{p2} & \dots & \xi_{pq} \end{pmatrix}, \quad \Xi \text{ is a } n \times n \text{ matrix.}$$

The restriction of the Segre family to the product of these Zariski open affine sets has the following canonical defining function:

$$\rho(z, \xi) = 1 + \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_k \leq p, \\ 1 \leq j_1 < j_2 < \dots < j_k \leq q \\ k=1, \dots, p}} Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \Xi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \quad (3)$$

Here $z = (z_{11}, z_{12}, \dots, z_{pq})$, $\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{pq})$. For simplicity of notation and terminology, we call this quasi-projective algebraic variety embedded in $\mathbb{C}^{pq} \times \mathbb{C}^{pq}$, which is defined by (3), the Segre family of $G(p, q)$.

♣2. Orthogonal Grassmannians (Type II Hermitian symmetric spaces of compact type): Write $G_{II}(n, n)$ for the submanifold of the Grassmannian $G(n, n)$ consisting of isotropic n -dimensional subspaces of \mathbb{C}^{2n} . Then $\tilde{S} \in G_{II}(n, n)$ if and only if

$$\tilde{S} \begin{pmatrix} 0 & I_{n \times n} \\ I_{n \times n} & 0 \end{pmatrix} \tilde{S}^T = 0. \quad (4)$$

In the aforementioned open affine piece of the Grassmannian $G(n, n)$ with $\tilde{S} = (I, S)$, $\tilde{S} \in G_{II}(n, n)$ if and only if S is an $n \times n$ antisymmetric matrix. We identify this open affine chart \mathcal{A}

of $G_{II}(n, n)$ with $\mathbb{C}^{\frac{n(n-1)}{2}}$ through the holomorphic coordinate map:

$$(I_{n \times n} \quad Z) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -z_{12} & 0 & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -z_{1n} & -z_{2n} & \cdots & 0 \end{pmatrix} \rightarrow (z_{12}, \dots, z_{(n-1)n}). \quad (5)$$

The Plücker embedding of $G(n, n)$ gives a 2-canonical embedding of $G_{II}(n, n)$. Unfortunately this embedding is not good enough for our purposes later. Therefore, we will use a different embedding in this paper, which is given by the spin representation of O_{2n} . This embedding is what is called a one-canonical embedding of $G_{II}(n, n)$. We briefly describe this embedding as following. More details can be found in [Chapter 12; PS].

Let V be a real vector space of dimension $2n$ with a given inner product, let $\mathcal{K}(V)$ be the space consisting of all orthogonal complex structures on V preserving this inner product. An element of $\mathcal{K}(V)$ is a linear orthogonal transformation $J : V \rightarrow V$ such that $J^2 = -1$. Any two choices of J are conjugate in the orthogonal group $O(V) = O_{2n}$, and thus $\mathcal{K}(V)$ can be identified with the homogeneous space O_{2n}/U_n . On the other hand, there is a one-to-one correspondence assigning the complex J to a complex n -dimensional isotropic subspace W of $V_{\mathbb{C}} (= V \otimes \mathbb{C})$. $\mathcal{K}(V)$ has two connected components $\mathcal{K}_{\pm}(V)$: Noticing that any complex structure defines an orientation on V , the two components correspond to the two possible orientations on V . Write one for $\mathcal{K}_+(V)$, which is actually our $G_{II}(n, n)$.

Now fix an isotropic n -dimensional subspace $W \subset V_{\mathbb{C}}$, with the associated complex structure J , of $V_{\mathbb{C}}$ and pick a basis for V : $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ with $J(x_i) = y_i, J(y_i) = -x_i$. Then W is spanned by $\{x_i - \sqrt{-1}y_i\}_{i=1}^n$. Define \overline{W} to be the space spanned by $\{x_i + \sqrt{-1}y_i\}_{i=1}^n$. As shown in [PS], there is a holomorphic embedding $\mathcal{K}(V) \hookrightarrow \mathbb{P}(\Lambda(W))$, where $\Lambda(W)$ is the exterior algebra of W . This embedding is equivariant under the action of $O(V)$. Thus $\mathcal{K}_+(V) \hookrightarrow \mathbb{P}(\Lambda(W))$ is equivariant under $SO(V)$. Choose the open affine cell of $\mathcal{K}_+(V)$ such that $\{Y \in \mathcal{K}_+(V) | Y \cap \overline{W} = \emptyset\}$. Then it can be identified with (5).

We next describe the 1-canonical embedding by Pfaffians as following:

Let Π be the set of all partitions of $\{1, 2, \dots, 2n\}$ into pairs without regard to order. There are $(2n-1)!!$ such partitions. An element $\alpha \in \Pi$ can be written as

$$\alpha = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < \dots < i_n$. Let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 2n \\ i_1 & j_1 & i_2 & j_2 & \dots & j_n \end{bmatrix}$$

be the corresponding permutation. Given a partition α as above and a $(2n) \times (2n)$ matrix $A = (a_{jk})$, define

$$A_{\alpha} = \text{sgn}(\pi) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}.$$

The Pfaffian of A is then given by

$$\text{pf}(A) = \sum_{\alpha \in \Pi} A_\alpha.$$

The Pfaffian of an $m \times m$ skew-symmetric matrix for m odd is defined to be zero.

Therefore in the coordinates system (5), the embedding is given by

$$[1, \dots, \text{pf}(Z_\sigma), \dots]. \quad (6)$$

Write S_k for the collection of all subsets of $\{1, \dots, n\}$ with k elements. The σ in (6) runs through all elements of S_k with $2 \leq k \leq n$ and k even. For instance, $(\text{pf}(Z_\sigma))_{\sigma \in S_2} = (z_{12}, \dots, z_{(n-1)n})$. We also write (6) as $[1, r_z] = [1, \psi_1, \psi_2, \dots, \psi_N]$ for simplicity of notation. We choose the local coordinates for $(G_{II}(n, n))^*$ in a similar way

$$(I_{n \times n} \quad \Xi) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \xi_{12} & \cdots & \xi_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -\xi_{12} & 0 & \cdots & \xi_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -\xi_{1n} & -\xi_{2n} & \cdots & 0 \end{pmatrix}. \quad (7)$$

The defining function for the Segre family (in the product of such affine pieces) is given by

$$\rho(z, \xi) = 1 + \sum_{\substack{\sigma \in S_k, \\ 2 \leq k \leq n, 2|k}} \text{Pf}(Z_\sigma) \text{Pf}(\Xi_\sigma) \quad (8)$$

♣3. Symplectic Grassmannians (Type III Hermitian symmetric spaces): Write $G_{III}(n, n)$ for the submanifold of the Grassmannian space $G(n, n)$ defined as follows: Take the matrix representation of each element of the Grassmannian $G(n, n)$ as an $n \times 2n$ non-degenerate matrix. Then $\tilde{A} \in G_{III}(n, n)$, if and only if,

$$\tilde{A} \begin{pmatrix} 0 & I_{n \times n} \\ -I_{n \times n} & 0 \end{pmatrix} \tilde{A}^T = 0 \quad (9)$$

In the Zariski open affine piece of the Grassmannian $G(n, n)$ defined before, we can take a representative matrix of the form: $\tilde{A} = (I, Z)$. Then we conclude that $\tilde{A} \in G_{III}(n, n)$ if and only if Z is an $n \times n$ symmetric matrix. We identify this Zariski open affine chart \mathcal{A} of $G_{III}(n, n)$ with $\mathbb{C}^{\frac{n(n+1)}{2}}$ through the holomorphic coordinate map:

$$\tilde{A} = (I_{n \times n} \quad Z) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & z_{11} & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & z_{12} & z_{22} & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & z_{1n} & z_{2n} & \cdots & z_{nn} \end{pmatrix} \rightarrow (z_{11}, \dots, z_{nn}).$$

Through the Plücker embedding of the Grassmannian $G(n, n)$, $G_{III}(n, n)$ is embedded into $\mathbb{P}(\Lambda^n \mathbb{C}^{2n})(\cong \mathbb{CP}^{N^*})$. In the above local coordinates, we write down the embedding as

$$Z \rightarrow [1, \dots, Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}, \dots] := [1, \psi_1, \dots, \psi_{N^*}]. \quad (10)$$

Choose the local affine open piece of $(G_{III}(n, n))^*$ consisting of elements in the following form:

$$(I_{n \times n} \quad \Xi) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \xi_{11} & \xi_{12} & \dots & \xi_{1n} \\ 0 & 1 & 0 & \dots & 0 & \xi_{12} & \xi_{22} & \dots & \xi_{2n} \\ & & & \dots & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & \xi_{1n} & \xi_{2n} & \dots & \xi_{nn} \end{pmatrix}.$$

The defining function of Segre family in the product of such affine open pieces is given by

$$\rho(z, \xi) = 1 + \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_k \leq n, \\ 1 \leq j_1 < j_2 < \dots < j_k \leq n \\ k=1, \dots, n}} Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \Xi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \quad (11)$$

However the Plücker embedding is not a useful canonical embedding to us for $G_{III}(n, n)$ due to the fact that $\{\psi_j\}$ is not a linearly independent system. For instance, we have the following relation:

$$Z \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + Z \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = Z \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

This embedding can not serve our purposes here. We therefore derive from this embedding a minimal embedding into a certain projective subspace in $\mathbb{P}(\Lambda^n \mathbb{C}^{2n})(\cong \mathbb{CP}^{N^*})$. We denote this minimal projective subspace by $\mathcal{H} \cong \mathbb{CP}^N$, which is discussed in detail below. We notice that the embedding $G_{III}(n, n) \hookrightarrow \mathbb{CP}^{N^*}$ is equivariant under the transitive action of $Sp(n)$.

Following the notations of the Grassmannians case (Type I), we write $[1, \psi_1, \dots, \psi_{N^*}]$ as the map of the Plücker embedding into \mathbb{CP}^{N^*} . Write $(\psi_{i_1}, \dots, \psi_{i_{m_k}})$ as its components of degree k in z . Here $1 \leq k \leq n$, and $\{i_1, \dots, i_{m_k}\}$ depends on k . For instance, if $k = 1$, then,

$$(\psi_{i_1}, \dots, \psi_{i_{m_1}}) = (z_{11}, \dots, z_{nn}),$$

where z_{ij} is repeated twice if $i \neq j$.

Let $\{\psi_1^{(k)}, \dots, \psi_{m_k^*}^{(k)}\}$ be a maximally linearly independent subset of $\{\psi_{i_1}, \dots, \psi_{i_{m_k}}\}$ over \mathbb{R} (and thus also over \mathbb{C}). For instance,

$$\{\psi_1^{(1)}, \dots, \psi_{m_1^*}^{(1)}\} = \{z_{ij}\}_{i \leq j}.$$

Let A_k as the $m_k^* \times m_k$ matrix such that $(\psi_{i_1}, \dots, \psi_{i_{m_k}}) = (\psi_1^*, \dots, \psi_{m_k^*}^*) \cdot A_k$. Apparently A_k has real entries and is of full rank. Hence $A_k \cdot A_k^t$ is positive definite.

Then $\{\psi_1^*, \dots, \psi_N^*\} := \{\psi_1^{(k)}, \dots, \psi_{m_k^*}^{(k)}\}_{1 \leq k \leq n}$ forms a basis of $\{\psi_1, \dots, \psi_{N^*}\}$, where $N = m_1^* + \dots + m_n^*$. Moreover, if we write A as the $(m_1^* + \dots + m_n^*) \times (m_1 + \dots + m_n)$ matrix:

$$A = \begin{pmatrix} A_1 & & \\ & \dots & \\ & & A_n \end{pmatrix},$$

Then A is full rank and we have a (real) orthogonal matrix U such that

$$U = \begin{pmatrix} U_1 & & \\ & \dots & \\ & & U_n \end{pmatrix}, \quad U^t(A \cdot A^t)U = \begin{pmatrix} \mu_1 & & \\ & \dots & \\ & & \mu_N \end{pmatrix} \quad \text{with all } \mu_j > 0.$$

Here $U_k, 1 \leq k \leq n$ is an $m_k^* \times m_k^*$ orthogonal matrix. Now we define

$$(\psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n) := (\psi_1^*, \dots, \psi_{N^*}^*) \cdot U \cdot \begin{pmatrix} \sqrt{\mu_1} & & & \\ & \sqrt{\mu_2} & & \\ & & \dots & \\ & & & \sqrt{\mu_N} \end{pmatrix}.$$

Here $N_1 + \dots + N_{n-1} + N_n = N$, where we set $N_n = 1$. We will also sometimes write $\psi_{N_n}^n = \psi^n$. As a direct consequence,

$$\begin{aligned} & (\psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n) \cdot (\overline{\psi_1^1}, \dots, \overline{\psi_{N_1}^1}, \overline{\psi_1^2}, \dots, \overline{\psi_{N_2}^2}, \dots, \overline{\psi_1^{n-1}}, \dots, \overline{\psi_{N_{n-1}}^{n-1}}, \overline{\psi^n}) \\ &= (\psi_1, \dots, \psi_{N^*}) \cdot (\overline{\psi_1}, \dots, \overline{\psi_{N^*}}) = \det(I + Z\bar{Z}^t) = \rho(z, \bar{z}). \end{aligned} \tag{12}$$

Moreover $\{\psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n\}$ forms a linearly independent system; and $\{\psi_1^k, \dots, \psi_{N_k}^k\}$ are polynomials in z of degree k for $k = 1, \dots, n$. And our canonical embedding of the aforementioned affine piece \mathcal{A} of $G_{III}(n, n)$ is taken as

$$z \in \mathbb{C}^{\frac{n(n+1)}{2}} \rightarrow [1, \psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n].$$

For simplicity, we will still denote $(\psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n)$ by

$$r_z = (\psi_1, \psi_2, \dots, \psi_N) = \left(\psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n \right). \tag{13}$$

Here for instance, $(\psi_1, \dots, \psi_{\frac{n(n+1)}{2}}) = (\psi_1^1, \dots, \psi_{N_1}^1) = (\delta_{ij} z_{ij})_{1 \leq i \leq j \leq n}$, where δ_{ij} equals to 1 if $i = j$, equals to $\sqrt{2}$ if $i < j$. Hence the defining function of the Segre family, which is the same as (11), is given by $\rho(z, \xi) = 1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi)$.

♣4. Hyperquadrics (Type IV Hermitian symmetric spaces): Let Q^n be the hypersurface in \mathbb{P}^{n+1} defined by

$$\left\{ [x_0, \dots, x_{n+1}] \in \mathbb{P}^{n+1} : \sum_{i=1}^n x_i^2 - 2x_0 x_{n+1} = 0 \right\},$$

where $[x_1, \dots, x_{n+2}]$ are the homogeneous coordinates for \mathbb{P}^{n+1} . It is invariant under the action of the group $SO(n+2)$. We mention that under the present embedding, the action is not the standard $SO(n+2)$ in $GL(n+2)$. However it is conjugate to the standard $SO(n+2)$ action by a certain element $g \in U(n+2)$. An Zariski open affine piece $\mathcal{A} \subset Q^n$ identified with \mathbb{C}^n is given by $(z_1, \dots, z_n) \mapsto [1, \psi_1, \dots, \psi_{n+1}] = [1, z_1, \dots, z_n, \frac{1}{2} \sum_{i=1}^n z_i^2]$, which will be denoted by $[1, r_z] = [1, \psi_1, \psi_2, \dots, \psi_{n+1}]$. Choose the same local chart for $(Q^n)^* : (\xi_1, \dots, \xi_n) \rightarrow [1, \xi_1, \dots, \xi_n, \frac{1}{2} \sum_{i=1}^n \xi_i^2]$. Then the defining function of the Segre family restricted to $\mathbb{C}^n \times \mathbb{C}^n \hookrightarrow Q^n \times (Q^n)^*$ is given by

$$\rho(z, \xi) = 1 + \sum_{i=1}^n z_i \xi_i + \frac{1}{4} \left(\sum_{i=1}^n z_i^2 \right) \left(\sum_{i=1}^n \xi_i^2 \right) \quad (14)$$

♣5. The exceptional manifold $M_{16} := E_6/SO(10) \times SO(2)$: As shown in [IM1],[IM2], this exceptional Hermitian symmetric space can be realize as the Cayley plane. Take the exceptional 3×3 complex Jordan algebra

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbb{C}, x_i \in \mathbb{O} \right\} \cong \mathbb{C}^{27} \quad (15)$$

Here \mathbb{O} is the complexified algebra of octonions, which is a complex vector space of dimension 8. Denote a standard basis of \mathbb{O} by $\{e_0, e_1, \dots, e_7\}$. The multiplication rule in terms of this basis is given in Appendix 2. The conjugation operator appeared in (15) is for octonions, which is defined as follows: $\bar{x} = x_0 e_1 - x_1 e_1 - \dots - x_7 e_7$, if $x = x_0 e_0 + x_1 e_1 + x_2 e_2 + \dots + x_7 e_7$, $x_i \in \mathbb{C}$.

The Jordan multiplication is defined as $A \circ B = \frac{1}{2}(AB + BA)$ for $A, B \in \mathcal{J}_3(\mathbb{O})$. The subgroup $SL(\mathbb{O})$ of $GL(\mathcal{J}_3(\mathbb{O}))$ consisting of automorphisms preserving the determinant is the adjoint group of type E_6 . The action of E_6 on the projectivization $\mathbb{P}\mathcal{J}_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface, the regular part of this hypersurface, and its singular part which is the closed E_6 -orbit. The closed orbit is the Cayley plane or the hermitian symmetric space of compact type corresponding to E_6 . It can be defined by the quadratic equation

$$X^2 = \text{trace}(X)X, \quad X \in \mathcal{J}_3(\mathbb{O}),$$

or as the closure of the affine cell \mathcal{A}

$$\mathbb{O}\mathbb{P}_1^2 = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & x\bar{x} & y\bar{x} \\ \bar{y} & x\bar{y} & y\bar{y} \end{pmatrix} : x, y \in \mathbb{O} \right\} \cong \mathbb{C}^{16}$$

in the local coordinates $(x_0, x_1, \dots, x_7, y_0, \dots, y_7)$. The precise formula for the canonical embedding map is given in Appendix 2. We denote this embedding by $[1, r_z] = [1, \psi_1, \psi_2, \dots, \psi_N]$.

To find the defining function for its Segre family over the product of such standard affine sets, we choose local coordinates for the conjugate Cayley plane to be $(\kappa_0, \kappa_1, \dots, \kappa_7, \eta_0, \eta_1, \dots, \eta_7)$. Then

$$\rho(z, \xi) = 1 + \sum_{i=0}^7 x_i \kappa_i + \sum_{i=0}^7 y_i \eta_i + \sum_{i=0}^7 A_i(x, y) A_i(\kappa, \eta) + B_0(x, y) B_0(\kappa, \eta) + B_1(x, y) B_1(\kappa, \eta), \quad (16)$$

where A_j, B_j are defined as in Appendix 2, $z = (x_0, \dots, x_7, y_0, \dots, y_7)$, and $\xi = (\kappa_0, \dots, \kappa_7, \eta_0, \dots, \eta_7)$.

♣6. The other exceptional manifold $M_{27} = E_7/E_6 \times SO(2)$: As shown in [CMP], it can be realized as the Freudenthal variety. Consider the Zorn algebra

$$\mathcal{Z}_2(\mathbb{O}) = \mathbb{C} \bigoplus \mathcal{J}_3(\mathbb{O}) \bigoplus \mathcal{J}_3(\mathbb{O}) \bigoplus \mathbb{C}$$

One can prove that there exists an action of E_7 on that 56-dimensional vector space (see [Fr]). The closed E_7 -orbit inside $\mathbb{P}\mathcal{Z}_2(\mathbb{O})$ is the Freudenthal variety $E_7/E_6 \times SO(2)$. An affine cell \mathcal{A} of Freudenthal variety is $[1, X, \text{Com}(X), \det(X)] \in \mathbb{P}\mathcal{Z}_2(\mathbb{O})$. Here X belongs to $\mathcal{J}_3(\mathbb{O})$ and its comatrix is defined by the usual formula for order three matrices such that $X\text{Com}(X) = \det(X)\text{I}$. For the explicit definition of $\text{Com}(X)$, see [O].

The embedding of $E_7/E_6 \times SO(2) \hookrightarrow \mathbb{P}^N$ in local coordinates z is given in Appendix 2. Choose the local affine open piece for $(E_7/E_6 \times SO(2))^*$ with coordinates

$$\xi = (\xi_1, \xi_2, \xi_3, \eta_0, \dots, \eta_7, \kappa_0, \dots, \kappa_7, \tau_0, \dots, \tau_7).$$

We denote this embedding by $[1, r_z] = [1, \psi_1, \psi_2, \dots, \psi_N]$. The defining function for the Segre family is then $\rho(z, \xi) = 1 + r_z \cdot r_\xi$, where

$$\begin{aligned} r_z &= (x_1, x_2, x_3, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7, A(z), B(z), C(z), D_0(z), \dots, D_7(z), \\ &\quad E_0(z), \dots, E_7(z), F_0(z), \dots, F_7(z), G(z)) \\ r_\xi &= (\psi_1(\xi), \psi_2(\xi), \dots, \psi_N(\xi)) = (\xi_1, \xi_2, \xi_3, \eta_0, \dots, \eta_7, \kappa_0, \dots, \kappa_7, \tau_0, \dots, \tau_7, \\ &\quad A(\xi), B(\xi), C(\xi), D_0(\xi), \dots, D_7(\xi), E_0(\xi), \dots, E_7(\xi), F_0(\xi), \dots, F_7(\xi), G(\xi)) \end{aligned} \tag{17}$$

Here see Appendix 2 for the definition of the functions appeared in the formula.

2.3 Explicit expressions of the volume forms

From now on, we assume that M is an irreducible Hermitian symmetric space of compact type and we choose the canonical embedding $M \hookrightarrow \mathbb{P}^N$ described in §2.2 according to its type. We denote the metric on M induced from Fubini-Study of \mathbb{P}^N by ω , and the volume form by $d\mu = \omega^n$ (up to a positive constant). Notice that the metric we obtained is always invariant under the action of a certain transitive subgroup $G \subset \text{Aut}(M)$. Hence by a theorem of Wolf [W], ω is the unique G invariant metric on M up to a scale. We claim ω must be Kähler-Einstein. Indeed, since the Ricci form $\text{Ric}(\omega)$ of ω is also invariant under G , as well, for a small ϵ , $\omega + \epsilon \text{Ric}(\omega)$ is also a G invariant metric on M . By [W], it is a multiple of ω , and thus $\text{Ric}(\omega) = \lambda \omega$. Write $d\mu$ as the multiplication of V and the standard Euclidean volume form, where V is a positive function in z . Since $\text{Ric}(\omega) = -i\partial\bar{\partial} \log V$, $-i\partial\bar{\partial} \log V = \lambda \omega$. Notice that $\lambda > 0$. In the local affine open piece \mathcal{A} defined before, $\omega = i\partial\bar{\partial} \log \rho(z, \bar{z})$, where $\rho(z, \xi)$ is the defining function for the associated Segre family. As we will see later (§7), $\rho(z, \xi)$ is an irreducible polynomial in (z, ξ) . Then we have

$$\partial\bar{\partial} \log(V\rho(z, \bar{z})^\lambda) = 0.$$

Hence, $\log(V\rho(z, \bar{z})^\lambda) = \phi(z) + \overline{\psi(z)}$, where both ϕ and ψ are holomorphic functions. Therefore $V = \frac{e^{\phi(z) + \overline{\psi(z)}}}{\rho(z, \bar{z})^\lambda}$. Because $\rho(z, \xi)$ is an irreducible polynomial, from the way V is defined, V must be a rational function of the form $\frac{p(z, \bar{z})}{\rho(z, \bar{z})^m}$ with p, q relatively prime to each other. Since ϕ, ψ are globally defined, by a monodromy argument, it is clear that λ has to be integer. Also both $e^{\phi(z)}$ and $e^{\overline{\psi(z)}}$ must be rational functions. Again, since ϕ, ψ are also globally defined, this forces ϕ, ψ to be constant functions. Therefore, we conclude that

$$V = c\rho(z, \bar{z})^{-\lambda}. \quad (18)$$

Here λ is a certain positive integer and c is a positive constant. Next by a well-known result (see [BaMa]), two Kähler-Einstein metrics of M are different by an automorphism of M . Therefore, to prove Theorem 1.1, we can assume, without loss of generality, that the Kähler-Einstein metric in Theorem 1.1 is the metric obtained by restricting the Fubini-Study metric to M through the embedding described in this section.

3 A basic property for degenerate holomorphic maps

In this section, we introduce a notion of degeneracy for holomorphic maps and derive an important consequence, which will be fundamentally applied in the proof of our main theorem.

Let $\psi(z) := (\psi_1(z), \dots, \psi_N(z))$ be a vector-valued holomorphic function from a neighborhood U of 0 in $\mathbb{C}^m, m \geq 2$, into $\mathbb{C}^N, N > m$, with $\psi(0) = 0$. Here we write $z = (z_1, \dots, z_m)$ for the coordinates of \mathbb{C}^m . In the following, we will write $\tilde{z} = (z_1, \dots, z_{m-1})$, i.e., the vector z with the last component z_m being dropped out. Write $\frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_{m-1}^{\alpha_{m-1}}}$ for an $(m-1)$ -multiindex α , where

$$\alpha = (\alpha_1, \dots, \alpha_{m-1}).$$

Write

$$\frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} \psi(z) = \left(\frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} \psi_1(z), \dots, \frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} \psi_N(z) \right).$$

We introduce the following definition.

Definition 3.1. Let $k \geq 0$. For a point $p \in U$, write $E_k(p) = \text{Span}_{\mathbb{C}}\{\frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} \psi(z)|_{z=p} : 0 \leq |\alpha| \leq k\}$. We write r for the greatest number such that for any neighborhood O of 0, there exists $p \in O$ with $\dim_{\mathbb{C}} E_k(p) = r$. r is called the k -th \tilde{z} -rank of ψ at 0, which is written as $\text{rank}_k(\psi, \tilde{z})$. F will be called \tilde{z} -nondegenerate if $\text{rank}_{k_0}(\psi, \tilde{z}) = N$ for some $k_0 \geq 1$.

Remark 3.2. It is easy to see that $\text{rank}_k(\psi) = r$ if and only if the following matrix

$$\begin{pmatrix} \frac{\partial^{|\alpha^0|}}{\partial \tilde{z}^{\alpha^0}} \psi(z) \\ \dots \\ \dots \\ \frac{\partial^{|\alpha^s|}}{\partial \tilde{z}^{\alpha^s}} \psi(z) \end{pmatrix}$$

has an $r \times r$ submatrix with determinant not identically zero for $z \in U$ for some multiindices $\{\alpha^0, \dots, \alpha^s\}$ with all $0 \leq |\alpha_j| \leq k$. Moreover, any $l \times l$ ($l > r$) submatrix of the matrix has identical zero determinant for any choice of $\{\alpha^0, \dots, \alpha^s\}$ with $0 \leq |\alpha_j| \leq k$.

In particular, ψ is \tilde{z} -nondegenerate if and only if there exist multiindices β^1, \dots, β^N such that

$$\begin{vmatrix} \frac{\partial |\beta^1|}{\partial \tilde{z}^{\beta^1}} \psi_1(z) & \dots & \frac{\partial |\beta^1|}{\partial \tilde{z}^{\beta^1}} \psi_N(z) \\ \dots & \dots & \dots \\ \frac{\partial |\beta^N|}{\partial \tilde{z}^{\beta^N}} \psi_1(z) & \dots & \frac{\partial |\beta^N|}{\partial \tilde{z}^{\beta^N}} \psi_N(z) \end{vmatrix}$$

is not identically zero.

Moreover, $\text{rank}_{i+1}(\psi, \tilde{z}) \geq \text{rank}_i(\psi, \tilde{z})$ for any $i \geq 0$.

In the following, we further assume that the first m components of ψ , i.e.,

$$(\psi_1, \dots, \psi_m) : \mathbb{C}^m \rightarrow \mathbb{C}^m$$

is a biholomorphic map in a neighborhood of $0 \in \mathbb{C}^m$. Then we have,

Lemma 3.3. $\text{rank}_0(\psi, \tilde{z}) = 1, \text{rank}_1(\psi, \tilde{z}) = m$. Hence, for any $k \geq 1, \text{rank}_k(\psi, \tilde{z}) \geq m$.

Proof of Lemma 3.3: We first note that it holds trivially that $\text{rank}_0(\psi, \tilde{z}) = 1$, for F is not identically zero. We now prove $\text{rank}_1(\psi, \tilde{z}) = m$. First note that $\text{rank}_1(\psi, \tilde{z}) \leq m$ as there are only m distinct multiindices β such that $|\beta| \leq 1$. On the other hand, since ψ has full rank at 0, we have,

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial z_1} & \dots & \frac{\partial \psi_m}{\partial z_1} \\ \dots & \dots & \dots \\ \frac{\partial \psi_1}{\partial z_m} & \dots & \frac{\partial \psi_m}{\partial z_m} \end{vmatrix} (0) \neq 0.$$

This together with the fact $\psi(0) = 0$ imply that the z_m derivative of

$$\begin{vmatrix} \psi_1 & \dots & \psi_m \\ \frac{\partial \psi_1}{\partial z_1} & \dots & \frac{\partial \psi_m}{\partial z_1} \\ \dots & \dots & \dots \\ \frac{\partial \psi_1}{\partial z_{m-1}} & \dots & \frac{\partial \psi_m}{\partial z_{m-1}} \end{vmatrix} \quad (19)$$

is nonzero at $p = 0$. Consequently, the quantity in (19) is not identically zero in U . We then arrive at the conclusion. ■

We now prove the following theorem, which will be crucially applied in our later discussions.

Theorem 3.4. Let $\psi = (\psi_1, \dots, \psi_m, \psi_{m+1}, \dots, \psi_N)$ be a holomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into \mathbb{C}^N . Assume as above that (ψ_1, \dots, ψ_m) is a biholomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into a neighborhood of $0 \in \mathbb{C}^m$. Suppose

$$\text{rank}_{N-m+1}(\psi, \tilde{z}) < N. \quad (20)$$

Then there exist N holomorphic functions $g_1(z_m), \dots, g_N(z_m)$ near 0 on the z_m -plane with $\{g_1(0), \dots, g_N(0)\}$ not all zero such that the following holds for any (z_1, \dots, z_m) near 0.

$$\sum_{i=1}^N g_i(z_m) \psi_i(z_1, \dots, z_m) \equiv 0. \quad (21)$$

In particular, one can make one of the $\{g_i\}_{i=1}^N$ to be identically one.

Proof of Theorem 3.4: We consider the following set,

$$\mathcal{S} = \{l \geq 1 : \text{rank}_l(\psi, \tilde{z}) \leq l + m - 2\}.$$

Note that $1 \notin \mathcal{S}$, for $\text{rank}_1(F) = m$. We claim that \mathcal{S} is not empty. Indeed, we have $1 + N - m \in \mathcal{S}$ by (20). Now set t' to be the minimum value of numbers in \mathcal{S} . Then $2 \leq t' \leq 1 + N - m$. Moreover, by the choice of t' ,

$$\text{rank}_{t'}(\psi, \tilde{z}) \leq t' + m - 2, \quad \text{rank}_{t'-1}(\psi, \tilde{z}) \geq t' + m - 2. \quad (22)$$

This yields that

$$\text{rank}_{t'}(\psi, \tilde{z}) = \text{rank}_{t'-1}(\psi, \tilde{z}) = t' + m - 2. \quad (23)$$

We write $t := t' - 1$, $n := t' + m - 2$. Here we note $t \geq 1, m \leq n \leq N - 1$. Then there exist multiindices $\{\gamma^1, \dots, \gamma^n\}$ with each $|\gamma^i| \leq t$ and j_1, \dots, j_n such that

$$\Delta(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n) := \begin{vmatrix} \frac{\partial^{|\gamma^1|} \psi_{j_1}}{\partial \tilde{z}^{\gamma^1}} & \dots & \frac{\partial^{|\gamma^1|} \psi_{j_n}}{\partial \tilde{z}^{\gamma^1}} \\ \dots & \dots & \dots \\ \frac{\partial^{|\gamma^n|} \psi_{j_1}}{\partial \tilde{z}^{\gamma^n}} & \dots & \frac{\partial^{|\gamma^n|} \psi_{j_n}}{\partial \tilde{z}^{\gamma^n}} \end{vmatrix} \text{ is not identically zero in } U. \quad (24)$$

Since $\text{rank}_1(\psi, \tilde{z}) = m$, we can choose $(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n)$ such that

$$\gamma^1 = (0, \dots, 0), \gamma^2 = (1, 0, \dots, 0), \dots, \gamma^m = (0, \dots, 0, 1).$$

For any $\alpha^1, \dots, \alpha^{n+1}$ with $|\alpha^i| \leq t + 1$, and l_1, \dots, l_{n+1} , we have

$$\Delta(\alpha^1, \dots, \alpha^{n+1} | l_1, \dots, l_{n+1}) = \begin{vmatrix} \frac{\partial^{|\alpha^1|} \psi_{l_1}}{\partial \tilde{z}^{\alpha^1}} & \dots & \frac{\partial^{|\alpha^1|} \psi_{l_n}}{\partial \tilde{z}^{\alpha^1}} & \frac{\partial^{|\alpha^1|} \psi_{l_{n+1}}}{\partial \tilde{z}^{\alpha^1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial^{|\alpha^{n+1}|} \psi_{l_1}}{\partial \tilde{z}^{\alpha^{n+1}}} & \dots & \frac{\partial^{|\alpha^{n+1}|} \psi_{l_n}}{\partial \tilde{z}^{\alpha^{n+1}}} & \frac{\partial^{|\alpha^{n+1}|} \psi_{l_{n+1}}}{\partial \tilde{z}^{\alpha^{n+1}}} \end{vmatrix} \equiv 0 \text{ in } U. \quad (25)$$

We write the set Γ to be the collection of $(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n)$, $j_1 < \dots < j_n$, with $\gamma^1 = (0, \dots, 0)$ such that (24) holds. We associate each $(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n)$ with an integer $s(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n) := s_0$ where s_0 is the least number $s \geq 0$ such that

$$\frac{\partial^{s_1 + \dots + s_{m-1} + s} \Delta(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n)}{\partial z_1^{s_1} \partial z_2^{s_2} \dots \partial z_{m-1}^{s_{m-1}} \partial z_m^s}(0) \neq 0.$$

for some integers s_1, \dots, s_{m-1} . Then $s(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n) \geq 0$ for any $(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n) \in \Gamma$.

Let $(\beta^1, \dots, \beta^n | i_1, \dots, i_n) \in \Gamma, i_1 < \dots < i_n$ be indices with the least $s(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n)$ among all $(\gamma^1, \dots, \gamma^n | j_1, \dots, j_n) \in \Gamma$.

We will need the following lemma proved in [BX]:

Lemma 3.5. ([BX], Lemma 4.4) *For a general $n \times n$ matrix*

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1n} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nn} \end{pmatrix},$$

where $b_{ij} \in \mathbb{C}$ for $1 \leq i, j \leq n, n \geq 3$, we have the following identity:

$$\begin{vmatrix} B \begin{pmatrix} 1 & 2 & \cdot & \cdot & \cdot & n-2 & n-1 \\ 1 & 2 & \cdot & \cdot & \cdot & n-2 & n-1 \end{pmatrix} & B \begin{pmatrix} 1 & 2 & \cdot & \cdot & \cdot & n-2 & n-1 \\ j_1 & j_2 & \cdot & \cdot & \cdot & j_{n-2} & n \end{pmatrix} \\ B \begin{pmatrix} i_1 & i_2 & \cdot & \cdot & \cdot & i_{n-2} & n \\ 1 & 2 & \cdot & \cdot & \cdot & n-2 & n-1 \end{pmatrix} & B \begin{pmatrix} i_1 & i_2 & \cdot & \cdot & \cdot & i_{n-2} & n \\ j_1 & j_2 & \cdot & \cdot & \cdot & j_{n-2} & n \end{pmatrix} \end{vmatrix} \quad (*)$$

$= B \begin{pmatrix} i_1 & i_2 & \cdot & \cdot & \cdot & i_{n-2} \\ j_1 & j_2 & \cdot & \cdot & \cdot & j_{n-2} \end{pmatrix} |B|$, for any $1 \leq i_1 < i_2 < \dots < i_{n-2} \leq n-1, 1 \leq j_1 < j_2 < \dots < j_{n-2} \leq n-1$. In particular, if $|B| = 0$, then $(*)$ equals 0. Here we have used the notation

$$B \begin{pmatrix} i_1 & i_2 & \cdot & \cdot & \cdot & i_p \\ j_1 & j_2 & \cdot & \cdot & \cdot & j_p \end{pmatrix} = \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & \cdot & \cdot & \cdot & b_{i_1 j_p} \\ b_{i_2 j_1} & b_{i_2 j_2} & \cdot & \cdot & \cdot & b_{i_2 j_p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{i_p j_1} & b_{i_p j_2} & \cdot & \cdot & \cdot & b_{i_p j_p} \end{vmatrix} \quad \text{for } 1 \leq p \leq n.$$

We write $\{i_{n+1}, \dots, i_N\} = \{1, \dots, N\} \setminus \{i_1, \dots, i_n\}$, where $i_{n+1} < \dots < i_N$. Write $\tilde{U} = \{z \in U : \Delta(\beta^1, \dots, \beta^n | i_1, \dots, i_n) \neq 0\}$. We then have the following:

Lemma 3.6. *Fix $j \in \{i_{n+1}, \dots, i_N\}$. Let $i \in \{i_1, \dots, i_n\}$. Write $\{i'_1, \dots, i'_{n-1}\} = \{i_1, \dots, i_n\} \setminus \{i\}$. There exists a holomorphic function $g_i^j(z_m)$ in \tilde{U} which only depends on z_m such that the following holds for $z \in \tilde{U}$:*

$$\begin{vmatrix} \frac{\partial |\beta^1| \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \dots & \frac{\partial |\beta^1| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial |\beta^1| \psi_j}{\partial \bar{z}^{\beta^1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial |\beta^n| \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \dots & \frac{\partial |\beta^n| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial |\beta^n| \psi_j}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) = g_i^j(z_m) \begin{vmatrix} \frac{\partial |\beta^1| \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \dots & \frac{\partial |\beta^1| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial |\beta^1| \psi_i}{\partial \bar{z}^{\beta^1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \frac{\partial |\beta^n| \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \dots & \frac{\partial |\beta^n| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial |\beta^n| \psi_i}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z), \quad (26)$$

or equivalently,

$$\begin{vmatrix} \frac{\partial^{|\beta^1|} \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|} \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial^{|\beta^1|} (\psi_j - g_i^j(z_{pq}) \psi_i)}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^n|} \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|} \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial^{|\beta^n|} (\psi_j - g_i^j(z_{pq}) \psi_i)}{\partial \bar{z}^{\beta^n}} \end{vmatrix} \equiv 0. \quad (27)$$

Proof of Lemma 3.6: For simplicity of notation, we write $\frac{\partial}{\partial \bar{z}^{\beta^i}}$ for $\frac{\partial^{|\beta^i|}}{\partial \bar{z}^{\beta^i}}$. To prove (26), one just needs to show that, for each $1 \leq \nu \leq m-1$,

$$\frac{\partial}{\partial z_\nu} \left(\begin{vmatrix} \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^n}} \\ \hline \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^n}} \end{vmatrix} \right) \equiv 0 \quad (28)$$

in \tilde{U} . Indeed, by the quotient rule, the numerator of the left-hand side of (28) equals to

$$\frac{\partial}{\partial z_\nu} \left(\begin{vmatrix} \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^n}} \\ \hline \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^n}} \end{vmatrix} \right) =$$

is a multiple of the determinant

$$\begin{vmatrix} \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^1}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^n}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^n}} \\ \frac{\partial \psi_{i'_1}}{\partial \bar{z}^{\beta^{n+1}}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^{n+1}}} & \frac{\partial \psi_i}{\partial \bar{z}^{\beta^{n+1}}} & \frac{\partial \psi_j}{\partial \bar{z}^{\beta^{n+1}}} \end{vmatrix}, \quad (30)$$

where $\frac{\partial}{\partial \bar{z}^{\beta^{n+1}}} = \frac{\partial}{\partial z_\nu} (\frac{\partial}{\partial \bar{z}^{\beta^n}})$, which is identically zero by (25). This establishes Lemma 3.6. ■

The extendability of $g_i^j(z_m)$ will be needed for our later argument, which is proved in the following:

Lemma 3.7. *For any i, j as above, the holomorphic function $g_i^j(z_m)$ can be extended holomorphically to a neighborhood of 0 in the z_m -plane.*

Proof of Lemma 3.7: First, g_i^j is defined on the projection $\pi_m(\tilde{U})$ of \tilde{U} , where π_m is the natural projection of (z_1, \dots, z_m) to its last component z_m . If $0 \in \pi_m(\tilde{U})$, then the claim is trivial. Now assume that $0 \notin \pi_m(\tilde{U})$. If we write $s = s(\beta_1, \dots, \beta_n | i_1, \dots, i_n)$, by its definition, then there exists $(a_1, \dots, a_{m-1}) \in \mathbb{C}^{m-1}$ close to 0, such that

$$\begin{vmatrix} \frac{\partial |\beta^1| \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial |\beta^1| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial |\beta^1| \psi_i}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial |\beta^n| \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial |\beta^n| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial |\beta^n| \psi_i}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (a_1, \dots, a_{m-1}, z_m) = cz_m^s + o(|z_m|^s), \quad c \neq 0. \quad (31)$$

Then there exists $r > 0$ small enough such that for any $0 < |z_m| < r$, $(a_1, \dots, a_{m-1}, z_m) \in \tilde{U}$. That is, at any of such points, equation (31) is not zero.

We now substitute $(a_1, \dots, a_{m-1}, z_m)$, $0 < |z_m| < r$, into the equation (26), and compare the vanishing order as $z_m \rightarrow 0$:

$$c_1 z_m^{s'} + o(|z_m|^{s'}) = g_i^j(z_m)(cz_m^s + o(|z_m|^s)), \quad c_1 \neq 0. \quad (32)$$

for some $s' \geq 0$. Note that $0 \leq s \leq s'$ by the definition of s and the choice of $(\beta_1, \dots, \beta_n | i_1, \dots, i_n)$. The holomorphic extendability across 0 of $g_i^j(z_m)$ then follows easily. ■

We next make the following observation:

Claim 3.8. *For each fixed $j \in \{i_{n+1}, \dots, i_N\}$ and any $i'_1 < \dots < i'_{n-1}$ with $\{i'_1, \dots, i'_{n-1}\} \subset \{i_1, \dots, i_n\}$, we have:*

$$\begin{vmatrix} \frac{\partial |\beta^1| \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial |\beta^1| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial |\beta^1| (\psi_j - \sum_{k=1}^n g_{i_k}^j \psi_{i_k})}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial |\beta^n| \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial |\beta^n| \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial |\beta^n| (\psi_j - \sum_{k=1}^n g_{i_k}^j \psi_{i_k})}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) \equiv 0, \quad \forall z \in \tilde{U}. \quad (33)$$

Proof of Claim 3.8: Note that for each $i'_l, 1 \leq l \leq n-1$, the following trivially holds:

$$\begin{vmatrix} \frac{\partial^{|\beta^1|} \psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|} \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \frac{\partial^{|\beta^1|} (g_{i'_l}^j \psi_{i'_l})}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^n|} \psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|} \psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \frac{\partial^{|\beta^n|} (g_{i'_l}^j \psi_{i'_l})}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) \equiv 0, \quad (34)$$

for the last column in the matrix is a multiple of one of the first $(n-1)$ columns. Then (33) is an immediate consequence of (27) and (34). ■

Lemma 3.9. *For each fixed $j \in \{i_{n+1}, \dots, i_N\}$, we have $\psi_j(z) - \sum_{k=1}^n g_{i_k}^j(z_m) \psi_{i_k}(z) \equiv 0$ for any $z \in \tilde{U}$, and thus it holds also for all $z \in U$.*

Proof of Lemma 3.9: This can be concluded easily from the following Lemma 3.10 and Claim 3.8. Here one needs to use the fact that $\beta^1 = (0, \dots, 0)$. ■

Lemma 3.10. *([BX], Lemma 4.7) Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ and \mathbf{a} be n -dimensional column vectors with elements in \mathbb{C} , and let $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ denote the $n \times n$ matrix. Assume that $\det B \neq 0$ and $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \dots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$ for any $1 \leq i_1 < i_2 < \dots < i_{n-1} \leq n$. Then $\mathbf{a} = \mathbf{0}$.*

Theorem 3.4 now follows easily from Lemma 3.9. ■

If we further assume that $\psi_i(z), m+1 \leq i \leq N$, vanishes at least to the second order, then we have the following.

Theorem 3.11. *Let $\psi = (\psi_1, \dots, \psi_m, \psi_{m+1}, \dots, \psi_N)$ be a holomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into \mathbb{C}^N . Assume that (ψ_1, \dots, ψ_m) is a biholomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into a neighborhood of $0 \in \mathbb{C}^N$. Assume that $\psi_j(z) = O(|z|^2)$ for $m+1 \leq j \leq N$. Suppose that $\text{rank}_{N-m+1}(\psi) < N$. Then there exist $a_{m+1}, \dots, a_N \in \mathbb{C}$ that are not all zero such that*

$$\sum_{j=m+1}^N a_j \psi_j(z_1, \dots, z_{m-1}, 0) \equiv 0, \quad (35)$$

for all (z_1, \dots, z_{m-1}) near 0.

Proof of Theorem 3.11: We first have the following:

Claim 3.12. *For each $1 \leq i \leq m$, $g_i(0) = 0$.*

Proof of Claim 3.12: Suppose not. Write $\mathbf{c} := (g_1(0), \dots, g_m(0)) \neq 0$. Then $(g_1(z), \dots, g_m(z)) = \mathbf{c} + O(|z|)$. The fact that $\psi_i(z) = O(|z|^2), i \geq m+1$, implies

$$\sum_{i=1}^m g_i(z) \psi_i(z) = O(|z|^2). \quad (36)$$

Note (ψ_1, \dots, ψ_m) is of full rank at 0. That is

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial z_1} & \cdots & \frac{\partial \psi_m}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_1}{\partial z_m} & \cdots & \frac{\partial \psi_m}{\partial z_m} \end{vmatrix} (0) \neq 0.$$

Hence

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial z_1}(0) & \cdots & \frac{\partial \psi_m}{\partial z_1}(0) \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_1}{\partial z_m}(0) & \cdots & \frac{\partial \psi_m}{\partial z_m}(0) \end{pmatrix} \mathbf{c}^t \neq 0. \quad (37)$$

This is a contradiction to (36). ■

Then we let $z_m = 0$ in equation (21) to obtain (35). Since

$$(g_1(0), \dots, g_m(0)) = 0, \text{ then } (g_{m+1}(0), \dots, g_N(0)) \neq 0.$$

This establishes Theorem 3.11.

■

4 Proof of the main theorem under three hypotheses

In this section, we give a proof of our main theorem under several assumptions which will be verified in the later sections.

Let $M \subset \mathbb{CP}^N$ be a Hermitian symmetric space of compact type, which has been canonically (and isometrically) embedded in the complex projective space through the way described in §2. In this section, we write n as the complex dimension of M . We also have on M an affine open \mathcal{A} that is biholomorphically equivalent to the complex Euclidean space of the same dimension, such that $M \setminus \mathcal{A}$ is a codimension one complex subvariety of M . We identify the coordinates of \mathcal{A} by the parametrization map with $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ through what is described in §2, which we wrote as $[1, \psi_1, \dots, \psi_N]$, where ψ_1, \dots, ψ_N are polynomial maps in (z_1, \dots, z_n) with $\psi_j = z_j$ for $j = 1, \dots, n$. We also write $\overline{F}(\xi)$ for $\overline{F(\xi)}$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$. We still use $\rho(z, \xi)$ for the defining function of the Segre family of M restricted to $\mathcal{A} \times \mathcal{A}^*$, which will be canonically identified with $\mathbb{C}^n \times \mathbb{C}^n$. Since the coefficients of ψ_1, \dots, ψ_N are all real, $\overline{\psi} = \psi$. Hence, we have

$$\rho(z, \xi) = 1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi). \quad (38)$$

Recall the standard metric ω of M on \mathcal{A} is given by

$$\omega = i\partial\bar{\partial}\log(\rho(z, \bar{z})). \quad (39)$$

The volume form $d\mu = c_n \omega^n$ associated to ω , by §2, is now given in \mathcal{A} by the multiplication of V and the standard Euclidean volume form, where

$$V = \frac{c}{(\rho(z, \bar{z}))^\lambda} \quad (40)$$

with $c > 0$ and λ a certain positive integer depending on M . (For instance, $\lambda = p + q$ when $X = G(p, q)$ [G]). Here c_n is a certain positive constant depending only on n .

Theorem 4.1. *Let $\mathcal{A} \subset M$ be as above equipped with the standard metric ω . Let $F_j, j = 1, \dots, m$, be a holomorphic map from $U \subset \mathcal{A}$ into M , where U is a connected open neighborhood of \mathcal{A} . Assume that $F_j^*(d\mu) \not\equiv 0$ for each j and assume that*

$$d\mu = \sum_{j=1}^m \lambda_j F_j^*(d\mu), \quad (41)$$

for certain positive constants $\lambda_j > 0$ with $j = 1, \dots, m$. Then for any $j \in \{1, 2, \dots, m\}$, F_j extends to a holomorphic isometry of (M, ω) .

For convenience of our discussions, we first fix some notations: In what follows, we identify \mathcal{A} with \mathbb{C}^n having $z = (z_1, \dots, z_n)$ as its coordinates. On $U \subset \mathcal{A} \subset M$ and after shrinking U if needed, we write the holomorphic map F_j , for $j = 1, \dots, m$, from $U \rightarrow \mathcal{A} = \mathbb{C}^n$, as follows:

$$F_j = (F_{j,1}, F_{j,2}, \dots, F_{j,n}), \quad j = 1, \dots, m. \quad (42)$$

Still write the holomorphic embedding from \mathcal{A} into \mathbb{P}^N as $[1, \psi_1, \dots, \psi_N]$. We define $\mathcal{F}_j(z) = (\mathcal{F}_{j,1}, \dots, \mathcal{F}_{j,N}) = (\psi_1(F_j), \psi_2(F_j), \dots, \psi_N(F_j))$ for $j = 1, \dots, m$. Finally, all Segre varieties and Segre families are restricted to $\mathcal{A} = \mathbb{C}^n$.

The main purpose of this section is to give the proof of Theorem 4.1 under the following three hypotheses. These hypotheses will be separately verified in terms of the type of M in §5, §6 and §7. This then completes the proof of our main theorem.

Hypothesis (I): Under the notations we just set up, for any $F_j, j = 1, \dots, m$, defined on $U \subset \mathbb{C}^n \subset M$, $\exists z^0 \in U, \xi^0 \in Q_{z^0}, \beta^1, \dots, \beta^N$, such that

$$\Lambda(\beta^1, \dots, \beta^N)(z^0, \xi^0) := \begin{vmatrix} \mathcal{L}^{\beta^1} \mathcal{F}_{j,1} & \dots & \mathcal{L}^{\beta^1} \mathcal{F}_{j,N} \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^N} \mathcal{F}_{j,1} & \dots & \mathcal{L}^{\beta^N} \mathcal{F}_{j,N} \end{vmatrix} (z^0, \xi^0) \quad (43)$$

is well-defined and non-zero. Here $\rho(z, \xi)$ is the defining function of the Segre family of $M \hookrightarrow \mathbb{CP}^N$ restricted to $\mathcal{A} \times \mathcal{A}^* = \mathbb{C}^n \times \mathbb{C}^n$; $\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z, \xi)}{\frac{\partial \rho}{\partial z_n}(z, \xi)} \frac{\partial}{\partial z_n}$, $0 \leq i \leq n-1$; $\beta^l = (k_1^l, \dots, k_{n-1}^l)$, k_1^l, \dots, k_{n-1}^l are non-negative integers, for $l = 1, 2, \dots, N$; $\beta^1 = (0, 0, \dots, 0)$; $\mathcal{L}^{\beta^l} = \mathcal{L}_1^{k_1^l} \mathcal{L}_2^{k_2^l} \mathcal{L}_3^{k_3^l} \dots \mathcal{L}_{n-1}^{k_{n-1}^l}$. Moreover, $s_l := \sum_{i=1}^{n-1} k_i^l$ ($l = 1, \dots, N$) is a non-negative integer bounded from above by a universal constant depending only on M .

Hypothesis (II): Suppose $\xi^0 \in \mathbb{C}^n, \xi^0 \neq (0, 0, \dots, 0)$. Then for a smooth point z^0 on the Segre variety Q_{ξ^0} and a small neighborhood $U \ni z^0$, there is a $z^1 \in U \cap Q_{\xi^0}$ such that Q_{z^0} and Q_{z^1} intersect transversally at ξ^0 .

Under this assumption, we can find a holomorphic parametrization near ξ^0 : $(\xi_1, \xi_2, \dots, \xi_n) = \mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n)$ with $(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n) \in U_1 \times U_2 \times \dots \times U_n \subset \mathbb{C}^n$, such that $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times \dots \times U_n) \subset Q_{z^0}$, $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times \dots \times U_n) \subset Q_{z^1}$, and $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times \dots \times U_n)$ or $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times \dots \times U_n)$, $s \in U_1, t \in U_2$ is an open piece of a certain Segre variety for each fixed t and s . Moreover \mathcal{G} consists of algebraic functions with total degree bounded by a constant depending only on the manifold M . (See Appendix 1).

Hypotheses (III): For any $\xi \in \mathbb{C}^n$, Q_ξ is irreducible. Moreover, if U is a connected open set in \mathbb{C}^n , then the Segre family \mathcal{M} restricted to $U \times \mathbb{C}^n$ is an irreducible complex subvariety and thus its regular points form a connected complex submanifold. Also, \mathcal{M} is an irreducible complex subvariety of $\mathbb{C}^n \times \mathbb{C}^n$.

The rest of this section will be divided into several subsections. In the first subsection, we discuss a partial algebraicity for a certain F_{j_0} among the maps in Theorem (4.1). In §5.2, we show F_{j_0} is algebraic. In §5.3, we further prove the rationality of F_{j_0} . §5.4 is devoted to proving that F_{j_0} extends to a birational map from M to itself and extends to a holomorphic isometry, which can be used, through an induction argument, to give a proof of Theorem (4.1) under the hypotheses in (I)-(III).

4.1 Algebraicity Lemma

We use the notations we have set up so far. We now proceed to the proof Theorem (4.1) under the hypotheses in (I)-(III).

From (38)(39)(40)(41), we obtain

$$\sum_{j=1}^m \lambda_j \frac{|J_{F_j}(z)|^2}{(1 + \sum_{i=1}^N \psi_i(F_j(z))\psi_i(\overline{F_j}(\bar{z})))^\lambda} = \frac{1}{(1 + \sum_{i=1}^N \psi_i(z)\psi_i(\bar{z}))^\lambda}, \quad z = (z_1, \dots, z_n) \in U. \quad (44)$$

Recall that $F_j = (F_{j,1}, F_{j,2}, \dots, F_{j,n}), j = 1, \dots, n$. Complexifying (44), we have

$$\sum_{j=1}^m \lambda_j \frac{J_{F_j}(z)\overline{J_{F_j}}(\xi)}{(1 + \sum_{i=1}^N \psi_i(F_j(z))\psi_i(\overline{F_j}(\xi)))^\lambda} = \frac{1}{(1 + \sum_{i=1}^N \psi_i(z)\psi_i(\xi))^\lambda}, \quad (z, \xi) \in U \times U^*. \quad (45)$$

Here U^* is the conjugation of U defined in §2.1. We will assume that (45) holds for $(z, \xi) \in U \times U$ and let $U = B_r(0)$ for a sufficiently small $r > 0$.

We will need the following algebraicity lemma.

Lemma 4.2. *Let F'_j s be as in Theorem 4.1. Then there exist Nash algebraic maps*

$$\hat{F}_1(z, X_1, \dots, X_m), \dots, \hat{F}_m(z, X_1, \dots, X_m)$$

holomorphic in (z, X_1, \dots, X_m) near $(0, \overline{J_{F_1}}(0), \dots, \overline{J_{F_m}}(0)) \in \mathbb{C}^n \times \mathbb{C}^m$ such that

$$\overline{F}_j(z) = \widehat{F}_j(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_m}}(z)), j = 1, \dots, m \quad (46)$$

for $z = (z_1, \dots, z_n)$ near 0.

Proof of Lemma 4.2: Recall that $\psi_j = z_j$ for $j = 1, \dots, n$ and $\psi_j = O(|z|^2)$ is a polynomial of z for each $n+1 \leq j \leq N$. We obtain from (45) the following:

$$\sum_{j=1}^m \lambda_j (J_{F_j}(z) \overline{J_{F_j}}(\xi) - \lambda (J_{F_j}(z) F_j(z)) \cdot (\overline{J_{F_j}}(\xi) \overline{F}_j(\xi)) + P_j(z, \overline{F}_j(\xi), \overline{J_{F_j}}(\xi))) = \frac{1}{(1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi))^\lambda}. \quad (47)$$

Here each

$$P_j(z, \overline{F}_j(\xi), \overline{J_{F_j}}(\xi))$$

is a rational function in its variables.

We now set $X_j = J_{F_j}$, $1 \leq j \leq m$. Set Y_j , $1 \leq j \leq m$, to be the vectors:

$$Y_j = (Y_{j1}, \dots, Y_{jn}) := J_{F_j} F_j = (J_{F_j} F_{j,1}, \dots, J_{F_j} F_{j,n}).$$

Then equation (47) can be rewritten as

$$\sum_{j=1}^m \lambda_j (X_j(z) \overline{X}_j(\xi) - \lambda Y_j(z) \cdot \overline{Y}_j(\xi) + Q_j(z, \overline{X}_j(\xi), \overline{Y}_j(\xi))) = \frac{1}{(1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi))^\lambda} \quad (48)$$

over $U \times U$. Here each Q_j with $1 \leq j \leq m$ is rational in $\overline{X}_j, \overline{Y}_j$. Moreover, each Q_j , $1 \leq j \leq m$, has no terms of the form $\overline{X}_j^k \overline{Y}_{js}^l$ with $l \leq 1$ for any $s \geq 1$ in its Taylor expansion at $(\overline{X}_j(0), \overline{Y}_j(0))$.

We write $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$ for any n -multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. Taking Differentiations in (48), we obtain, for each multiindex α ,

$$\sum_{j=1}^m ((D^\alpha X_j(z)) \overline{X}_j(\xi) - \lambda (D^\alpha Y_j(z)) \cdot \overline{Y}_j(\xi) + D^\alpha Q_j(z, \overline{X}_j(\xi), \overline{Y}_j(\xi))) = D^\alpha \left(\frac{1}{(1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi))^\lambda} \right) \quad (49)$$

Again each $D^\alpha Q_j$, $1 \leq j \leq m$, is rational in $(\overline{X}_j, \overline{Y}_j)$ and has no terms of the form $\overline{X}_j^k \overline{Y}_{js}^l$ with $l \leq 1$ and $s \geq 1$ in its Taylor expansion at $(\overline{X}_j(0), \overline{Y}_j(0))$. Applying a similar argument as in [Proposition 3.1, HY], we arrive at the conclusion.

■

Let \mathcal{R} be the field of rational functions in $z = (z_1, \dots, z_n)$. Consider the field extension

$$\mathcal{E} = \mathcal{R}(\overline{J_{F_1}}(z), \dots, \overline{J_{F_m}}(z)).$$

Let K be the transcendental degree of the field extension \mathcal{E}/\mathcal{R} . If $K = 0$, then each of $\{\overline{J_{F_1}}, \dots, \overline{J_{F_m}}\}$ is Nash algebraic. As a consequence of Lemma 4.2, each $F_j, 1 \leq j \leq m$ is Nash algebraic. Otherwise, by re-ordering the indices if necessary, we let $\mathcal{G} = \{\overline{J_{F_1}}, \dots, \overline{J_{F_K}}\}$ be the maximal algebraic independent subset of $\{\overline{J_{F_1}}, \dots, \overline{J_{F_m}}\}$. It follows that the transcendental degree of $\mathcal{E}/\mathcal{R}(\mathcal{G})$ is zero. For any $l > K$, there exists a minimal polynomial $P_l(z, X_1, \dots, X_K, X)$ such that $P_l(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z), \overline{J_{F_l}}(z)) \equiv 0$. Moreover,

$$\frac{\partial P_l(z, X_1, \dots, X_K, X)}{\partial X}(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z), \overline{J_{F_l}}(z)) \neq 0$$

in a small neighborhood V of 0, for otherwise, P_l cannot be a minimal polynomial of $\overline{J_{F_l}}(z)$. Now the vanishing of the partial derivatives for all l forms a proper local complex analytic variety near 0. Applying the algebraic version of the implicit function theorem, there exists a small connected open subset $U_0 \subset U$, with $0 \in \overline{U_0}$ and a holomorphic algebraic function $\widehat{h}_l, l > K$, in the neighborhood \widehat{U}_0 of $\{(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z)) : z \in U_0\}$ in $\mathbb{C}^n \times \mathbb{C}^K$, such that

$$\overline{J_{F_l}}(z) = \widehat{h}_l(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z)),$$

for any $z \in U_0$. (We can assume here U_0 is the projection of \widehat{U}_0). Substitute this into

$$\widehat{F}_i(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_m}}(z)),$$

and still denote it, for simplicity of notation, by $\widehat{F}_j(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z))$ with

$$\widehat{F}_j(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z)) = \widehat{F}_j(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_m}}(z)) \text{ for } z \in U_0.$$

In the following, for simplicity of notation, we also write for $j \leq K$,

$$\widehat{h}_j(z, \overline{J_{F_1}}(z), \dots, \overline{J_{F_K}}(z)) = \overline{J_{F_j}}(z) \text{ or } \widehat{h}_j(z, X_1, \dots, X_K) = X_j.$$

Now we replace $\overline{F_j}(\xi)$ by $\widehat{F}_j(\xi, \overline{J_{F_1}}(\xi), \dots, \overline{J_{F_K}}(\xi))$, and replace $\overline{J_{F_j}}(\xi)$ by $\widehat{h}_j(\xi, \overline{J_{F_1}}(\xi), \dots, \overline{J_{F_K}}(\xi))$, for $1 \leq j \leq m$, in (45).

Furthermore, we write $X = (X_1, \dots, X_K)$, and replace $\overline{J_{F_j}}(\xi)$ by X_j for $1 \leq j \leq K$ in

$$\widehat{F}_j(\xi, \overline{J_{F_1}}(\xi), \dots, \overline{J_{F_K}}(\xi)), \widehat{h}_j(\xi, \overline{J_{F_1}}(\xi), \dots, \overline{J_{F_K}}(\xi)), 1 \leq j \leq m.$$

We define a new function Φ as follows:

$$\Phi(z, \xi, X) := \sum_{j=1}^m \lambda_j \frac{J_{F_j}(z) \widehat{h}_j(\xi, X)}{(1 + \sum_{i=1}^N \psi_i(F_j(z)) \psi_i(\widehat{F}_j(\xi, X)))^\lambda} - \frac{1}{(1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi))^\lambda}. \quad (50)$$

Lemma 4.3. *Shrinking U if necessary, we have $\Phi(z, \xi, X) \equiv 0$, i.e.,*

$$\sum_{j=1}^m \lambda_j \frac{J_{F_j}(z) \widehat{h}_j(\xi, X)}{(1 + \sum_{i=1}^N \psi_i(F_j(z)) \psi_i(\widehat{F}_j(\xi, X)))^\lambda} = \frac{1}{(1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi))^\lambda}. \quad (51)$$

or,

$$\begin{aligned}
& \left(1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi)\right)^\lambda \sum_{j=1}^m \left(\lambda_j J_{F_j}(z) \widehat{h}_j(\xi, X) \prod_{1 \leq k \leq m, k \neq j} \left(1 + \sum_{i=1}^N \psi_i(F_j(z)) \psi_i(\widehat{F}_j(\xi, X))\right)^\lambda \right) \\
&= \prod_{1 \leq k \leq m} \left(1 + \sum_{i=1}^N \psi_i(F_j(z)) \psi_i(\widehat{F}_j(\xi, X))\right)^\lambda
\end{aligned} \tag{52}$$

for any $z \in U$ and $(\xi, X) \in \widehat{U}_0$.

Proof of Lemma 4.3: We use a similar argument as in [HY1]. Suppose not. Notice Φ is Nash algebraic in (ξ, X) by Lemma 4.2. For a generic fixed $z = z_0$ near 0, since $\Phi(z, \xi, X) \not\equiv 0$, there exist polynomials $A_l(\xi, X)$ for $0 \leq l \leq N$ with $A_0(\xi, X) \not\equiv 0$ such that

$$\sum_{0 \leq l \leq N} A_l(\xi, X) \Phi^l(z, \xi, X) \equiv 0.$$

As $\Phi(z_0, \xi, \overline{J_{F_1}(\xi)}, \dots, \overline{J_{F_K}(\xi)}) \equiv 0$ for $\xi \in U_0$, then it follows that $A_0(\xi, \overline{J_{F_1}(\xi)}, \dots, \overline{J_{F_K}(\xi)}) \equiv 0$ for $\xi \in U_0$. This is a contradiction to the assumption that $\{\overline{J_{F_1}(\xi)}, \dots, \overline{J_{F_K}(\xi)}\}$ is an algebraic independent set.

■

Now that $\widehat{F}_j(\xi, X)$, $1 \leq j \leq m$, is algebraic in its variables, if \widehat{F}_j , $1 \leq j \leq m$, is independent of X , then F_j is algebraic by Lemma 4.2. This fact motivates the remaining work in this section.

4.2 Algebraicity and rationality with bounded degree

In this subsection, we prove the algebraicity and rationality for at least one of the F'_j s. We start with the following:

Lemma 4.4. *Let $F_j(z)$, $j \in \{1, \dots, m\}$, be a local holomorphic map defined on a neighborhood of $0 \in U$ as in (4.1). Suppose that there exist $z^0 \in U$ and $\xi^0 \in Q_{z^0}$ such that $\Lambda(\beta^1, \dots, \beta^N)(z^0, \xi^0)$ is well defined and non-zero with $\beta^1 = (0, 0, \dots, 0)$. Then there is an analytic variety $W \subsetneq U$, such that when $z \in U \setminus W$, $\Lambda(\beta^1, \dots, \beta^N)(z, \xi)$ is well-defined over Q_z as a rational function in ξ and $\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \not\equiv 0$ on Q_z .*

Proof of Lemma 4.4: By the assumption,

$$\Lambda(\beta^1, \dots, \beta^N)(z^0, \xi^0) = \begin{vmatrix} \mathcal{L}^{\beta^1} \mathcal{F}_{j,1} & \dots & \mathcal{L}^{\beta^1} \mathcal{F}_{j,N} \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^N} \mathcal{F}_{j,1} & \dots & \mathcal{L}^{\beta^N} \mathcal{F}_{j,N} \end{vmatrix} (z^0, \xi^0) \tag{53}$$

is well defined and non-zero with $\beta^1 = (0, 0, \dots, 0)$.

By the definition, $\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z, \xi)}{\frac{\partial \rho}{\partial z_n}(z, \xi)} \frac{\partial}{\partial z_n}$ and $\mathcal{L}^{\beta^l} = \mathcal{L}_1^{k_1^l} \mathcal{L}_2^{k_2^l} \mathcal{L}_3^{k_3^l} \dots \mathcal{L}_{n-1}^{k_{n-1}^l}$. Hence $\Lambda(\beta^1, \dots, \beta^N)(z, \xi)$ can be written in the form $\Lambda(\beta^1, \dots, \beta^N)(z, \xi) = \frac{\mathcal{G}_1(z, \xi)}{\mathcal{G}_2(z, \xi)}$. Here $\mathcal{G}_1(z, \xi) = \sum_{|I|=0}^{M_1} \Phi_I(z) \xi^I$, $\mathcal{G}_2(z, \xi) = \sum_{|J|=0}^{M_2} \Psi_J(z) \xi^J$, Φ_I and Ψ_J are holomorphic functions defined on $U \subset \mathbb{C}^n$.

Since $\Lambda(\beta^1, \dots, \beta^N)(z^0, \xi^0) \neq 0$, we can assume $\mathcal{G}_1, \mathcal{G}_2$ are not zero at (z^0, ξ^0) . Hence, $\mathcal{G}_1, \mathcal{G}_2$ are non zero elements in $\mathcal{O}(U)[\xi_1, \dots, \xi_n]$, the polynomial ring of ξ with coefficients from the holomorphic function space over U .

By Hypothesis (III), the defining function of the Segre family ρ can be written in the form $\rho(z, \xi) = \sum_{|\alpha|=0}^{M_3} \Theta_\alpha(z) \xi^\alpha$, which is an irreducible polynomial in (z, ξ) . And for each fixed z , by Hypothesis (III), we also have $\rho(z, \xi)$ irreducible as a polynomial of ξ only.

Then the set of $z \in U$ where $\Lambda(\beta^1, \dots, \beta^N)(z, \xi)$ is undefined over Q_z is a subset of $z \in U$ where $\mathcal{G}_2(z, \xi)$, as a polynomial of ξ , contains the factor $\rho(z, \xi)$. We denote the latter set by W_2 . Similarly, the set of $z \in U$ with $\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \equiv 0$ over Q_z is a subset of $z \in U$ where $\mathcal{G}_1(z, \xi)$, as a polynomial of ξ , contains a factor $\rho(z, \xi)$, which we denote by W_1 .

Notice that $\rho(z, \xi) \in \mathcal{O}(U)[\xi_1, \dots, \xi_n]$ depends on each ξ_j for $1 \leq j \leq n$. If $\mathcal{G}_2(z, \xi)$ depends only on z , then $W_2 = \{\mathcal{G}_2(z) = 0\}$. Assume that $\mathcal{G}_2(z, \xi)$ depends on some ξ_i , say ξ_1 .

We next characterize W_2 by the resultant R_2 of $\mathcal{G}_2(z, \xi)$ and $\rho(z, \xi)$ as polynomials in ξ_1 . We rewrite \mathcal{G}_2 and ρ as polynomials of ξ_1 as follows:

$$\mathcal{G}_2 = \sum_{i=0}^k a_i(z, \xi_2, \dots, \xi_n) \xi_1^i, \quad \rho = \sum_{j=0}^l b_j(z, \xi_2, \dots, \xi_n) \xi_1^j.$$

Here the leading terms $a_k, b_l \neq 0$ with $k, l \geq 1$. We write the resultant as $R_2(z, \xi_2, \dots, \xi_n) = \sum_I c_I(z) \xi^I$, where c_I 's are holomorphic functions of $z \in U$.

For those points $z \in W_2$, $R_2(z, \cdot) \equiv 0$ as a polynomial of ξ_2, \dots, ξ_n . Then W_2 is contained in the complex analytic set $\widetilde{W}_2 := \{c_I = 0, \forall I\}$. If $\widetilde{W}_2 = U$, then we can find non-zero polynomials $f, g \in \mathcal{O}(U)[\xi_2, \dots, \xi_n][\xi_1]$ such that $f\rho + g\mathcal{G}_2 \equiv 0$, where the degree of g in ξ_1 is less than the degree of ρ in ξ_1 . Hence $\{\mathcal{G}_2 = 0\} \cup \{g = 0\} \supset \{\rho = 0\} \cap (U \times \mathbb{C}^n)$. Again by the irreducibility of $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$, since $\{g = 0\}$ is a thin set in $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$, \mathcal{G}_2 vanishes on $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$. This contradicts $\mathcal{G}_2(z^0, \xi^0) \neq 0$. Hence $W_2 \subset \widetilde{W}_2$ and \widetilde{W}_2 is a proper complex analytic set of U .

By the same argument, we can prove that W_1 is contained in \widetilde{W}_1 that is also a proper analytic set of U . Let $W = \widetilde{W}_1 \cup \widetilde{W}_2$. Then when $z \in U \setminus W$, $\Lambda(\beta^1, \dots, \beta^N)(z, \xi)$ is well-defined over Q_z as a rational function in ξ and $\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \neq 0$ on Q_z .

■

Lemma 4.5. *Let $\psi(\xi, X)$ be a Nash-algebraic function in $(\xi, X) = (\xi_1, \dots, \xi_n, X_1, \dots, X_m) \in \mathbb{C}^n \times \mathbb{C}^m$. Write E for a proper complex analytic variety of $\mathbb{C}^n \times \mathbb{C}^m$ that contains its branch locus and the zeros of its leading coefficient in its minimal polynomial. Then there exists a*

proper analytic set W_1 in \mathbb{C}^n such that

$$\{\xi \mid \exists X^0, (\xi, X^0) \notin E\} \supset \mathbb{C}^n \setminus W_1.$$

Proof of Lemma 4.5: Since ψ is algebraic, there is an irreducible polynomial $\Phi(\xi, X; Y) = \sum_{i=0}^k \phi_i(\xi, X)Y^i$ such that $\Phi(\xi, X, \psi(\xi, X)) \equiv 0$. If $k = 1$ then ψ is a rational function and thus E is just the poles and points of indeterminacy. We hence assume $k \geq 2$.

Define $\Psi(\xi, X, Y) = \frac{\partial \Phi}{\partial Y}$. Since $k \geq 2$, the degree of Ψ in Y is at least one. Consider Φ, Ψ as polynomials in Y , and write $R(\xi, X)$ for their resultant. Then the branch locus is contained in $\{(\xi, X) \mid R(\xi, X) = 0\}$. Notice that $R \not\equiv 0$, for Φ is irreducible. Write $R = \sum_I r_I(\xi)X^I$ with not all $r_I(\xi)$ identically zero. Write $\phi_k(\xi, X) = \sum \phi_{k,i}(\xi)X^i$ and $W_1 = \{r_I(\xi) = 0, \forall I\} \cup \{\phi_{k,i}(\xi) = 0, \forall i\}$, which is a proper complex analytic set in \mathbb{C}^n . Then $\{\xi \mid \exists X^0, (\xi, X^0) \notin E\} \supset \mathbb{C}^n \setminus W_1$.

■

Let E be the union of the branch loci of \hat{h}_j, \hat{F}_j for $j = 1, \dots, m$ and the zeros of the leading coefficients in their minimal polynomials. For any point $(z^0, \xi^0, X^0) \in U \times ((\mathbb{C}^n \times \mathbb{C}^m) \setminus E)$, we can find a smooth simple loop γ in $U \times ((\mathbb{C}^n \times \mathbb{C}^m) \setminus E)$ connecting (z^0, ξ^0, X^0) and a point in $U \times (\hat{U}_0 \setminus E)$. We can holomorphically continue the following equation along γ :

$$\begin{aligned} & (1 + \sum_{i=1}^N \psi_i(z)\psi_i(\xi))^\lambda \sum_{j=1}^m \lambda_j J_{F_j}(z) \hat{h}_j(\xi, X) \prod_{1 \leq k \leq m, k \neq j} (1 + \sum_{i=1}^N \psi_i(F_j(z))\psi_i(\hat{F}_j(\xi, X)))^\lambda \\ &= \prod_{1 \leq k \leq m} (1 + \sum_{i=1}^N \psi_i(F_j(z))\psi_i(\hat{F}_j(\xi, X)))^\lambda, \quad z \in U, (\xi, X) \in \hat{U}_0, \end{aligned} \tag{54}$$

to a neighborhood of (z^0, ξ^0, X^0) .

For our later discussions, we further define

$$\mathcal{M}_{\text{sing}, z} = \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}, \mathcal{M}_{\text{reg}, z} = \mathcal{M} \setminus \mathcal{M}_{\text{sing}, z};$$

$$\mathcal{M}_{\text{SING}} = \{(z, \xi) : \frac{\partial \rho}{\partial \xi_j} = 0, \forall j\} \cup \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}, \quad \mathcal{M}_{\text{REG}} = \mathcal{M} \setminus \mathcal{M}_{\text{SING}};$$

$\text{Pr}_z : \mathbb{C}^{2n} \rightarrow \mathbb{C}^n \quad (z, \xi) \mapsto (z)$ and $\text{Pr}_\xi : \mathbb{C}^{2n} \rightarrow \mathbb{C}^n \quad (z, \xi) \mapsto (\xi)$. Notice that \mathcal{M}_{REG} is a Zariski open subset of \mathcal{M} and the restrictions of $\text{Pr}_z, \text{Pr}_\xi$ to \mathcal{M}_{REG} are open mappings. Also, for $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}}$, Q_{z^0} is smooth at ξ^0 , and Q_{ξ^0} is smooth at z^0 .

Lemma 4.6. *Under the notations we have set up so far, there exists a point $(z^0, \xi^0, X^0) \in (U \times \mathbb{C}^n \times \mathbb{C}^m)$ with $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}} \cap (U \times \mathbb{C}^n)$ and $(\xi^0, X^0) \notin E$. Moreover, for each $j = 1, \dots, m$, we can find $\beta_j^1, \dots, \beta_j^N$ with $\beta_j^1 = (0, \dots, 0)$ such that $\Lambda(\beta^1, \dots, \beta^N)(z^0, \xi^0) \neq 0$.*

Proof of Lemma 4.6: This is an easy consequence of Hypothesis (I), Hypothesis (III) and (4.4). ■

Now, we analytically continue the equation (54) to a neighborhood of the point (z^0, ξ^0, X^0) obtained in Lemma (4.6). We denote one of such neighborhoods by $V_1 \times V_2 \times V_3$, where V_1, V_2 and V_3 are chosen to be a small neighborhood of z^0, ξ^0 , and X^0 , respectively. It is clear that there exists a $j_0 \in \{1, \dots, m\}$ such that

$$1 + \sum_{i=1}^N \psi_i(F_{j_0}(z)) \psi_i(\widehat{F_{j_0}}(\xi, X)) = 0, \quad \text{for } (z, \xi) \in \mathcal{M} \cap (V_1 \times V_2).$$

We next proceed to prove the algebraicity for $F_{j_0}(z)$.

Theorem 4.7. *$\widehat{F_{j_0}}(\xi, X)$ for $\xi \in V_2$, $X \in V_3$ is independent of X and is thus a Nash algebraic function of ξ . Hence F_{j_0} is an algebraic function of z . Moreover, the algebraic total degree of $\widehat{F_{j_0}}(\xi, X) = \overline{F_{j_0}}(\xi)$, and thus of $F_{j_0}(z)$, is uniformly bounded by a constant depending only on the manifold (X, ω) .*

Proof of Theorem 4.7: By the choice of (z^0, ξ^0, X^0) , there exist $\beta_{j_0}^1, \dots, \beta_{j_0}^N$ such that

$$\Lambda(\beta_{j_0}^1, \dots, \beta_{j_0}^N)(z^0, \xi^0) = \begin{vmatrix} \mathcal{L}^{\beta_{j_0}^1} \mathcal{F}_{j_0,1} & \dots & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N} \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,1} & \dots & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N} \end{vmatrix} (z^0, \xi^0) \neq 0. \quad (55)$$

By Hypothesis (II), we can find $z^1 \in V_1 \cap Q_{\xi^0}$, such that Q_{z^0} intersects Q_{z^1} transversally at ξ^0 . Therefore there exists a neighborhood B of ξ^0 and a biholomorphic parametrization of B : $(\xi_1, \xi_2, \dots, \xi_n) = \mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n)$ with $(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n) \in U_1 \times U_2 \times \dots \times U_n \subset \mathbb{C}^n$. Here U_j 's are small neighborhood of $1 \in \mathbb{C}$. Moreover, $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times \dots \times U_n) \subset Q_{z^0}$, $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times \dots \times U_n) \subset Q_{z^1}$. Also, for each $s \in U_1, t \in U_2$ $\mathcal{G}(\{\tilde{\xi}_0 = t\} \times U_2 \times \dots \times U_n)$, $\mathcal{G}(U_1 \times \{\tilde{\xi}_1 = s\} \times U_3 \times \dots \times U_n)$ is an open piece of a certain Segre variety. Here \mathcal{G} consists of algebraic functions with total algebraic degree uniformly bounded by M .

Consider the equation:

$$1 + \mathcal{F}_{j_0}(z) \cdot \widehat{\mathcal{F}_{j_0}}(\xi, X) = 0, \quad (z, \xi, X) \in V_1 \times V_2 \times V_3, (z, \xi) \in \mathcal{M}. \quad (56)$$

Since the vector fields $\{\mathcal{L}_i\}_{i=1}^{n-1}$ are tangential to the Segre family, we have

$$\begin{pmatrix} \mathcal{L}^{\beta_{j_0}^1} \mathcal{F}_{j_0,1}(z, \xi) & \dots & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N}(z, \xi) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,1}(z, \xi) & \dots & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N}(z, \xi) \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{F}_{j_0,1}}(\xi, X) \\ \dots \\ \widehat{\mathcal{F}_{j_0,N}}(\xi, X) \end{pmatrix} = \begin{pmatrix} -1 \\ \dots \\ 0 \end{pmatrix}, \quad (57)$$

where $(z, \xi)(\approx (z^0, \xi^0)) \in \mathcal{M}$, $X \approx X^0$.

By the Cramer's rule, we conclude that $\{\widehat{\mathcal{F}}_{j_0,l}(\xi, X)\}_{l=1}^N$ are rational functions of ξ with a uniformly bounded degree on an open piece of each Segre variety Q_z for $z \approx z^0$.

By a modified Hurwitz theorem as proved in Appendix 1, we conclude the algebraicity of $\widehat{\mathcal{F}}_{j_0,l}(\xi, X)$ for $l = 1, \dots, N$. Since in (57), the matrix and the right hand side are independent of X , these functions must also be independent of the X -variable. Moreover, by Lemma A.3 and Theorem A.4 in Appendix 1, the total algebraic degree of $\overline{F}_{j_0,l}(\xi) = \widehat{\mathcal{F}}_{j_0,l}(\xi, X)$, for $l = 1, \dots, n$, is uniformly bounded. Since \overline{F} is obtained by holomorphically continuing the conjugation function \overline{F} of F , we conclude the algebraicity of $F_{j_0,l}$ for each $1 \leq l \leq n$. Also the total algebraic degree of each $F_{j_0,l}$ is bounded by a constant depending only on (M, ω) .

■

Theorem 4.8. *Under the notations we have just set up, F_{j_0} is a rational map, whose degree depends only on the canonical embedding $M \hookrightarrow \mathbb{CP}^N$.*

For the proof Theorem (4.8), we first recall the following Lemma from [HZ]:

Lemma 4.9. *Let $U \subset \mathbb{C}^n$ be a simply connected open subset and $\mathcal{S} \subset U$ be a closed complex analytic subset of codimension one. Then for $p \in U \setminus \mathcal{S}$, the fundamental group $\pi_1(U \setminus \mathcal{S}, p)$ is generated by loops obtained by concatenating (Jordan) paths $\gamma_1, \gamma_2, \gamma_3$, where γ_1 connects p with a point arbitrarily close to a smooth point $q_0 \in \mathcal{S}$, γ_2 is a loop around \mathcal{S} near q_0 and γ_3 is γ_1 reversed.*

Proof of Theorem 4.8: We give the proof of the rationality for F_{j_0} . Once this is done, the degree of F_{j_0} is uniformly bounded, for we know the total algebraic degree of F is uniformly bounded by Theorem 4.7.

Suppose that F_{j_0} and thus \overline{F}_{j_0} is not rational. We choose a sufficiently small neighborhood W of (z^0, ξ^0) in \mathcal{M}_{REG} such that for each $(z^1, \xi^1) \in W$, we can find a loop of the form $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$ in $\mathbb{C}^n \setminus E$ with $\gamma(0) = \gamma(1) = \xi^1, \gamma_1(1) = q$. Here E is a proper complex analytic variety containing the branch locus of $F_{j_0}, \overline{F}_{j_0}$ and the zeros of the leading coefficients of the minimal polynomials of their components; γ_1 is a simple curve connecting ξ^1 to q with q in a small ball B_p centered at a smooth point p of E such that the fundamental group of $B_p \setminus E$ is generated by γ_2 ; and γ_1^{-1} is the reverse curve of γ_1 . Moreover, when \overline{F}_{j_0} is holomorphically continued along γ , we end up with a different branch $\overline{F}_{j_0}^*$ of \overline{F}_{j_0} . We pick p such that there is an $X_p \notin E$ with $(X_p, p) \in \mathcal{M}_{\text{reg}, z}$. Make B_p sufficiently small such that it is compactly contained in the image of the projection of a certain neighborhood \mathcal{W} of (X_p, p) in $\mathcal{M}_{\text{reg}, z}$. Also \mathcal{W} , without loss of generality, is defined by a holomorphic function of the form $z_1 = \phi(z_2, \dots, z_n, \xi)$. Write $X_q = (\phi(z_2^p, \dots, z_n^p, q), z_2^p, \dots, z_n^p)$ and define the loop $\gamma_2^*(t) = (\phi(z_2^p, \dots, z_n^p, \gamma_2(t)), z_2^p, \dots, z_n^p)$. Then γ_2^* is a loop with the base point at X_q with $(\gamma_2(t), \gamma_2^*(t)) \in \mathcal{M}$.

Notice that $X_p \notin E$. After shrinking B_p if needed, we assume that γ_2^* is homotopically trivial in $\mathbb{C}^n \setminus E$.

Now we slightly thicken γ_1 to get a simply connected domain U_1 of $\mathbb{C}^n \setminus E$. Since \mathcal{M} is irreducible over $\mathbb{C}^n \times U_1$, we can find a smooth simple curve $\tilde{\gamma}_1 = (\gamma_1^*, \hat{\gamma}_1)$ in \mathcal{M} connecting

(z^1, ξ^1) to (X_q, q) . Then $\widehat{\gamma}_1$ is homotopic to γ_1 relatively to $\{\xi^1, q\}$ and $\gamma_1^*(1) = X_q$. Now replace γ by its homotopically equivalent loop $\widehat{\gamma}_1^{-1} \circ \gamma_2 \circ \widehat{\gamma}_1$ and define $\gamma^* = \gamma_1^{*-1} \circ \gamma_2^* \circ \gamma_1^*$. Define $\Gamma = (\gamma^*, \gamma)$. Then $\Gamma \subset \mathcal{M} \setminus ((E \times \mathbb{C}^n) \cup (\mathbb{C}^n \times E))$. Continuing Equation (56) along Γ and noticing that it is independent of X now, we get

$$1 + \mathcal{F}_{j_0}(z) \cdot \overline{\mathcal{F}_{j_0}}^*(\xi) = 0, \quad (z, \xi) \in \mathcal{M} \cap ((V_1 \setminus E) \times (V_2 \setminus E)).$$

Now argue as before and apply the uniqueness of the linear system with invertible coefficient matrix. We conclude that $\overline{F_{j_0}}^* \equiv \overline{F_{j_0}}$. This is a contradiction.

■

4.3 Isometric extension of F

For simplicity of notation, in the rest of this section, we denote the map F_{j_0} just by F . Now that all components of F are rational functions, it is easy to verify that F gives rise to a rational map $M \dashrightarrow M$. By the Hironaka theorem (see [H] and [K]), we have a complex manifold Y , holomorphic maps $\tau : Y \rightarrow M$, $\sigma : Y \rightarrow M$, and a proper complex analytic variety E_1 of M such that $\sigma : Y \setminus \sigma^{-1}(E_1) \rightarrow M \setminus E_1$ is biholomorphic; $F : M \setminus E_1 \rightarrow M$ is well-defined; and for any $p \in Y \setminus \sigma^{-1}(E_1)$, $F(\sigma(p)) = \tau(p)$.

Let E_2 be a proper complex analytic subvariety of M containing E_1 , $M \setminus \mathcal{A}$ and the points where F fails to be locally biholomorphic. Write $E^* = \tau(\sigma^{-1}(E_2))$ and $E = \sigma(\tau^{-1}(E^*))$. Then $F : \mathcal{A} \setminus E \rightarrow \mathcal{A} \setminus E^*$ is a holomorphic covering map. We first prove

Lemma 4.10. : *Under the above notation, $F : \mathcal{A} \setminus E \rightarrow \mathcal{A} \setminus E^*$ is a biholomorphic map.*

Proof of Lemma 4.10: We first notice that since F is biholomorphic near 0 with $F(0) = 0$. We can assume that $0 \notin E$. Consider $F^2 = F \circ F$. Then $\overline{F^2} = \overline{F}^2$. Since (F, \overline{F}) maps \mathcal{M} into \mathcal{M} whenever it is defined, it is easy to see that $(F, \overline{F}) \circ (F, \overline{F}) = (F^2, \overline{F}^2)$ also maps \mathcal{M} into \mathcal{M} at the points where it is well-defined. Hence, we can apply the argument for F to conclude that F^2 , as a rational map, also has its degree bounded by a constant independent of F^2 . Similarly, we can conclude that for any positive integer m , F^m is a rational function map with degree bounded by a constant independent of m and F . Now, as for F , we can find complex analytic subvarieties $E^{(m)}$, $E^{*(m)}$ of \mathbb{C}^n such that F^m is a holomorphic covering from $\mathcal{A} \setminus E^{(m)} \rightarrow \mathcal{A} \setminus E^{*(m)}$. Suppose $F : \mathcal{A} \setminus E \rightarrow \mathcal{A} \setminus E^*$ is a k to 1 covering. It is easy to see that $F^m : \mathcal{A} \setminus E^{(m)} \rightarrow \mathcal{A} \setminus E^{*(m)}$ is a k^m to 1 covering. However, since the degree F^m is independent of m , we conclude that $k = 1$ by the following Bezout theorem:

Theorem 4.11. ([S]) The number of isolated solutions to a system of polynomial equations

$$f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0$$

is bounded by $d_1 d_2 \cdots d_n$, where $d_i = \deg f_i$.

This proves the lemma.

■

Now we prove that F extends to a global holomorphic isometry of (M, ω) .

Theorem 4.12. $F : (U, \omega|_U) \rightarrow (M, \omega)$ extends to a global holomorphic isometry of (M, ω) .

Proof of Theorem 4.12: By what we just achieved, we then have two proper complex analytic varieties W_1, W_2 of \mathbb{C}^n such that $F : \mathbb{C}^n \setminus W_1 \rightarrow \mathbb{C}^n \setminus W_2$ is biholomorphic. Similarly we have two proper complex analytic subvarieties W_1^*, W_2^* of \mathbb{C}^n such that $\overline{F} : \mathbb{C}^n \setminus W_1^* \rightarrow \mathbb{C}^n \setminus W_2^*$ is a biholomorphic map. Hence

$$\mathfrak{F} = (F, \overline{F}) : \mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^* \rightarrow \mathbb{C}^n \setminus W_2 \times \mathbb{C}^n \setminus W_2^*$$

is a biholomorphic.

Let ρ be the defining function of the Segre family as described before. Since ρ is irreducible as a polynomial in (z, ξ) , \mathcal{M} is an irreducible complex analytic variety of \mathcal{A} . Since \mathfrak{F} maps a certain open piece of \mathcal{M} into an open piece of \mathcal{M} , by the uniqueness of holomorphic functions, we see that $\mathfrak{F} = (F, \overline{F})$ also gives a biholomorphic map from $(\mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^*) \cap \mathcal{M}$ to $(\mathbb{C}^n \setminus W_2 \times \mathbb{C}^n \setminus W_2^*) \cap \mathcal{M}$. Hence $\rho_F = \rho(F(z), \overline{F}(\xi))$ defines the same subvariety as ρ does over $\mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^*$. Since ρ_F is a rational function in (z, ξ) with denominator coming from the factors of the denominators of $F(z)$ and $\overline{F}(\xi)$, we can write

$$\rho_F(z, \xi) = (\rho(z, \xi))^l \frac{P_1^{i_1}(z, \xi) P_2^{i_2}(z, \xi) \cdots P_\lambda^{i_\lambda}(z, \xi)}{Q_1^{j_1}(z) \cdots Q_\mu^{j_\mu}(z) R_1^{k_1}(\xi) \cdots R_\nu^{k_\nu}(\xi)} \quad (58)$$

Here the zeros of $Q_j(z)$ and $R_j(\xi)$ stay in W_1 and W_1^* , respectively. All polynomials are irreducible and prime to each other. By what we just mentioned $P_j(z, \xi)$ can not have any zeros in $\mathbb{C}^n \setminus W_1 \times \mathbb{C}^n \setminus W_1^*$, for otherwise it must have ρ as its factor by the irreducibility of ρ . Hence the zeros of $P_j(z, \xi)$ must stay in $(W_1 \times \mathbb{C}^n) \cup (\mathbb{C}^n \times W_1^*)$. From this, it follows easily that $P_j(z, \xi) = P_{j,1}(z)$ or $P_j(z, \xi) = P_{j,2}(\xi)$. Namely, $P_j(z, \xi)$ depends either on z or on ξ . Since \mathfrak{F} is a biholomorphic, $l = 1$. Thus replace ξ by \bar{z} and taking $i\partial\bar{\partial}\log$ to (58), we have $i\partial\bar{\partial}\log \rho_F(z, \bar{z}) = i\partial\bar{\partial}\log \rho(z, \bar{z})$. This shows that $F^*(\omega) = \omega$, or F is a local isometry. Now, by the Calabi theorem, F extends to a global holomorphic isometry of (M, ω) . This proves Theorem 4.12.

■

We now are ready to give a proof of Theorem 4.1. By what we have obtained, there is a component F_j for F in Theorem 4.1 that extends to a holomorphic isometry to (M, ω) . Hence $F_j^*(d\mu) = d\mu$. Notice $\lambda_j < 1$ due to the positivity of all terms in the right hand side of the equation (41). After a cancelation, we reduce the theorem to the case with only $(m-1)$ - maps. Then by induction, we complete the proof of Theorem 4.1.

5 Verification of Hypothesis (I)

In this long section, we verify the statement in Hypothesis (I) for compact Hermitian spaces of compact type. Since the argument differs in terms of their types, we start with the Grassmannian space.

5.1 Type I spaces

With the same notations that we have set up in §2, Z is a $p \times q$ matrix ($p \leq q$); $Z \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ is the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, \dots, i_k^{\text{th}}$ rows and $j_1^{\text{th}}, \dots, j_k^{\text{th}}$ columns; $z = (z_{11}, \dots, z_{1q}, z_{21}, \dots, z_{2q}, \dots, z_{p1}, \dots, z_{pq})$ is the coordinates of $\mathbb{C}^{pq} \cong \mathcal{A} \subset G(p, q)$.

For convenience of our discussions, we represent the map $F_j : U \rightarrow \mathcal{A}$, $j = 1, \dots, m$, as a holomorphic matrix-valued map:

$$F_j = \begin{pmatrix} f_{j,11} & \cdots & f_{j,1q} \\ \cdots & \cdots & \cdots \\ f_{j,p1} & \cdots & f_{j,pq} \end{pmatrix}, \quad j = 1, \dots, m.$$

Similar to $Z \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$, $F_j \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ denotes the determinant of the submatrix formed by the $i_1^{\text{th}}, \dots, i_k^{\text{th}}$ rows and $j_1^{\text{th}}, \dots, j_k^{\text{th}}$ columns of the matrix F_j , $j = 1, \dots, m$.

Recall the definition in (2):

$$r_z = (\psi_1, \psi_2, \dots, \psi_N) = (\cdots, Z \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}, \cdots), \quad 1 \leq i_1 < \cdots < i_k \leq p, 1 \leq j_1 < \cdots < j_k \leq q, 1 \leq k \leq p.$$

Similarly, we define:

$$r_{F_j} := (\cdots, F_j \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}, \cdots), \quad j = 1, \dots, m, 1 \leq i_1 < \cdots < i_k \leq p, 1 \leq j_1 < \cdots < j_k \leq q, 1 \leq k \leq p.$$

Notice that $r_{F_j} = (\psi_1(F_j(z)), \dots, \psi_N(F_j(z))), j = 1, \dots, m$. We define $\tilde{z} := (z_{11}, \dots, z_{1q}, z_{21}, \dots, z_{2q}, \dots, z_{p1}, \dots, z_{p(q-1)})$, i.e., \tilde{z} is obtained from z by dropping the last component z_{pq} . Write $\frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} = \frac{\partial^{|\alpha|}}{\partial z_{11}^{\alpha_{11}} \cdots \partial z_{p(q-1)}^{\alpha_{p(q-1)}}}$ for any $(pq-1)$ -multiindex α , where

$$\alpha = (\alpha_{11}, \dots, \alpha_{1p}, \alpha_{21}, \dots, \alpha_{2q}, \dots, \alpha_{p1}, \dots, \alpha_{p(q-1)}).$$

We apply the notion of the degeneracy defined in Definition 3.1 of Section 3: We let $\psi = r_{F_j}$ and let \tilde{z} be as just defined with $m = pq$.

We now prove the following proposition:

Proposition 5.1. *r'_{F_j} s are \tilde{z} -nondegenerate near 0. More precisely, $\text{rank}_{1+N-pq}(r_{F_j}, \tilde{z}) = N$, $j = 1, \dots, m$.*

Proof of Proposition 5.1: Suppose not. Without loss of generality, we can assume

$$\text{rank}_{1+N-pq}(r_{F_1}, \tilde{z}) < N.$$

It is easy to verify that the hypothesis of Theorem 3.11 is satisfied. Consequently, we see that there exist $c_{pq+1}, \dots, c_N \in \mathbb{C}$ which are not all zero such that

$$\sum_{i=pq+1}^N c_i \psi_i(F_1)(z_{11}, \dots, z_{pq-1}, 0) \equiv 0. \quad (59)$$

The next step is to show that (59) cannot hold in the setting of Proposition 5.1. This can be done by the following

Lemma 5.2. *Let*

$$H = \begin{pmatrix} h_{11} & \dots & h_{1p} \\ \dots & \dots & \dots \\ h_{p1} & \dots & h_{pq} \end{pmatrix},$$

be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (z_{11}, \dots, z_{p(q-1)}) \in \mathbb{C}^{pq-1}$ with $H(0) = 0$. Assume that H is of full rank at 0. Set

$$(\phi_1, \dots, \phi_m) := r_H = \left(\left(H \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \right)_{1 \leq i_1 < \dots < i_k \leq p, 1 \leq j_1 < \dots < j_k \leq q} \right)_{2 \leq k \leq p}. \quad (60)$$

Here

$$m = \binom{p}{2} \binom{q}{2} + \dots + \binom{p}{p} \binom{q}{p}.$$

Let a_1, \dots, a_m be complex numbers such that

$$\sum_{i=1}^m a_i \phi_i(\tilde{z}) \equiv 0 \text{ for all } \tilde{z} \in U. \quad (61)$$

Then

$$a_i = 0$$

for all $1 \leq i \leq m$.

Proof of Lemma 5.2: We start with the simple case $p = q = 2$, in which $m = 1$. Then by the assumption (61), $a_1 \phi_1 = 0$. Here

$$\phi_1 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}.$$

Note that $H = (h_{11}, h_{12}, h_{21}, h_{22})$ is of full rank at 0. We assume, without loss of generality, that $\tilde{H} := (h_{11}, h_{12}, h_{21})$ is a local biholomorphic map from \mathbb{C}^3 to \mathbb{C}^3 . After an appropriate biholomorphic change of coordinates preserving 0, we can assume $h_{11} = z_{11}, h_{12} = z_{12}, h_{21} = z_{21}$, and still write the last component as h_{22} . Then we have

$$a_1\phi_1 = a_1(z_{11}h_{22} - z_{12}z_{21}) \equiv 0,$$

which easily yields that $a_1 = 0$.

We then prove the lemma for the case of $p = 2, q = 3$, in which $m = 3$. As before, without loss of generality, we assume that $\tilde{H} := (h_{11}, h_{12}, h_{13}, h_{21}, h_{22})$ is a biholomorphic map from \mathbb{C}^5 to \mathbb{C}^5 . After an appropriate biholomorphic change of coordinates, we assume that $\tilde{H} = (z_{11}, \dots, z_{22})$. By (61), we have

$$a_1\phi_1 + \dots + a_3\phi_3 = a_1 \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} + a_2 \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & h_{23} \end{vmatrix} + a_3 \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & h_{23} \end{vmatrix}. \quad (62)$$

The conclusion can be easily proved by checking the coefficients in the Taylor expansion at 0. Indeed, the quadratic terms $z_{13}z_{21}, z_{13}z_{22}$ only appear once in last two determinants. This implies $a_2 = a_3 = 0$. Then trivially $a_1 = 0$.

We also prove the case $p = q = 3$. In this case $m = 10$. As before, without loss of generality, we assume that $\tilde{H} = (h_{11}, \dots, h_{32})$ is a biholomorphic map from \mathbb{C}^8 to \mathbb{C}^8 . After an appropriate biholomorphic change of coordinates, we can assume that $\tilde{H} = (z_{11}, \dots, z_{32})$. Then by assumption, we have,

$$\begin{aligned} a_1\phi_1 + \dots + a_{10}\phi_{10} = & a_1 \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} + a_2 \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} + a_3 \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} + a_4 \begin{vmatrix} z_{11} & z_{12} \\ z_{31} & z_{32} \end{vmatrix} + a_5 \begin{vmatrix} z_{11} & z_{13} \\ z_{31} & h_{33} \end{vmatrix} + a_6 \begin{vmatrix} z_{12} & z_{13} \\ z_{32} & h_{33} \end{vmatrix} \\ & + a_7 \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} + a_8 \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & h_{33} \end{vmatrix} + a_9 \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & h_{33} \end{vmatrix} + a_{10} \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & h_{33} \end{vmatrix} = 0. \end{aligned} \quad (63)$$

We then check the coefficients for each term in its Taylor expansion at 0. First it is easy to note that $a_5 = a_6 = a_8 = a_9 = 0$ by checking the coefficients of quadratic terms

$$z_{13}z_{31}, z_{13}z_{32}, z_{23}z_{31}, z_{23}z_{32},$$

respectively. Then by checking the coefficients of other quadratic terms, we see that $a_1 = a_2 = a_3 = a_4 = a_7 = 0$. Finally we check the coefficient of the cubic term $z_{13}z_{22}z_{31}$ to obtain that $a_{10} = 0$.

We now prove for the general case $q \geq p \geq 2$. As before, we assume without loss of generality that $\tilde{H} = (h_{11}, \dots, h_{p(q-1)})$ is a biholomorphic map from \mathbb{C}^{pq-1} to \mathbb{C}^{pq-1} . Furthermore, we have $\tilde{H} = (z_{11}, \dots, z_{p(q-1)})$ after an appropriate biholomorphic change of coordinates. We again first

consider the coefficients of the quadratic terms in the assumption (61). For that, we consider the 2×2 submatrix involving h_{pq} , i.e., $H \begin{pmatrix} l & p \\ k & q \end{pmatrix}$, $1 \leq l < p, 1 \leq k < q$. Note that $z_{lq}z_{pk}$ only appears in this 2×2 determinant, which yields that the coefficient a_i associated to this 2×2 determinant is 0, for all $1 \leq i < p, 1 \leq j < q$. Then by checking the coefficients of other quadratic terms, we see that all coefficients a_i that are associated to 2×2 determinants $H \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix}$, $1 \leq l_1, l_2 \leq p, 1 \leq k_1, k_2 \leq q$, are 0.

We then consider the coefficients of cubic terms in (61). We first look at those 3×3 submatrix involving h_{pq} , i.e., $H \begin{pmatrix} l_1 & l_2 & p \\ k_1 & k_2 & q \end{pmatrix}$, $1 \leq l_1 < l_2 < p, 1 \leq k_1 < k_2 < q$. Note that $z_{l_1q}z_{l_2k_2}z_{pk_1}$ only appears in this 3×3 matrix, which yields that the a_i associated to this 3×3 determinant is 0. Furthermore, we see that all coefficients a_i 's that are associated to 3×3 determinants are 0.

Then the conclusion can be proved inductively. Indeed, assume that we have proved that all coefficients a_i 's that are associated the determinants of order up to $\mu \times \mu$, $3 \leq \mu < p$. Now we prove that the coefficients associated to $(\mu + 1) \times (\mu + 1)$ determinants are 0. For this we consider all such determinants which involve h_{pq} , i.e., $H \begin{pmatrix} l_1 & \dots & l_\mu & p \\ k_1 & \dots & k_\mu & q \end{pmatrix}$ where $1 \leq l_1 < \dots < l_\mu < p, 1 \leq k_1 < \dots < k_\mu < q$. We conclude the a_i associated to it is 0 by noting that $z_{l_1q}z_{l_2k_2} \dots z_{l_\mu k_\mu}z_{pk_1}$ only appears in this $(\mu + 1) \times (\mu + 1)$ determinant. Then we can show all coefficients that are associated to other $(\mu + 1) \times (\mu + 1)$ determinants, i.e.,

$$H \begin{pmatrix} l_1 & \dots & l_\mu & l_{\mu+1} \\ k_1 & \dots & k_\mu & k_{\mu+1} \end{pmatrix}, 1 \leq l_1 < \dots < l_{\mu+1} \leq p, 1 \leq k_1 < \dots < k_{\mu+1} \leq q, (l_{\mu+1}, k_{\mu+1}) \neq (p, q).$$

are 0 by checking the term $z_{l_1k_1} \dots z_{l_{\mu+1}k_{\mu+1}}$ which only appears once in the Taylor expansion. This establishes the Lemma.

■

We thus get a contradiction to the equation (59). This establishes Proposition 5.1.

■

Remark 5.3. Let F be any of F_1, \dots, F_m . By Proposition 5.1, there exist multiindices β^1, \dots, β^N , with all $|\beta^j| \leq 1 + N - pq$, and

$$Z^0 = \begin{pmatrix} z_{11}^0 & \dots & z_{1q}^0 \\ \dots & \dots & \dots \\ z_{p1}^0 & \dots & z_{pq}^0 \end{pmatrix},$$

near 0, such that

$$\Delta(\beta^1, \dots, \beta^N) := \begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \bar{z}^{\beta^1}} & \dots & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \bar{z}^{\beta^1}} \\ \dots & \dots & \dots \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \bar{z}^{\beta^N}} & \dots & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \bar{z}^{\beta^N}} \end{vmatrix} (z^0) \neq 0. \quad (64)$$

Perturbing z^0 if necessary, we can assume that $z_{pq}^0 \neq 0$. Moreover, by (3.10), we can replace one of β^1, \dots, β^N by $\beta = (0, \dots, 0)$, because $(\psi_1(F), \dots, \psi_N(F))$ are not identically zero. Without loss of generality, we can assume $\beta^1 = (0, \dots, 0)$.

The defining function of the Segre family is

$$\rho(z, \xi) = 1 + \sum_{k=1}^p \left(\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq p, 1 \leq j_1 < j_2 < \dots < j_k \leq q} Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \Xi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \right) \quad (65)$$

It is a complex manifold for any fixed ξ close enough to the point

$$\xi^0 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \xi_{pq}^0 \end{pmatrix} \in \mathbb{C}^{pq},$$

where $\xi_{pq}^0 = -\frac{1}{z_{pq}^0}$.

Write for each $1 \leq i \leq p, 1 \leq j \leq q, (i, j) \neq (p, q)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z, \xi)}{\frac{\partial \rho}{\partial z_{pq}}(z, \xi)} \frac{\partial}{\partial z_{pq}}, \quad (66)$$

which is holomorphic tangent vector field along \mathcal{M} near (z^0, ξ^0) . Here we note that $\frac{\partial \rho}{\partial z_{pq}}(z, \xi)$ is nonzero near (z^0, ξ^0) . For any $(pq-1)$ -multiindex $\beta = (\beta_{11}, \dots, \beta_{p(q-1)})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{11}^{\beta_{11}} \dots \mathcal{L}_{p(q-1)}^{\beta_{p(q-1)}}.$$

Now we define for any N collection of $(pq-1)$ -multiindices $\{\beta^1, \dots, \beta^N\}$,

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix} (z, \xi). \quad (67)$$

We have the following,

Theorem 5.4. *There exists multiindices $\{\beta^1, \dots, \beta^N\}$, such that*

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \neq 0, \quad (68)$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) . Moreover, we can require $\beta^1 = (0, \dots, 0)$.

Proof of Theorem 5.4: First we observe that \mathcal{L}_{ij} evaluating at (z^0, ξ^0) is just $\frac{\partial}{\partial z_{ij}}$. More generally, for any $(pq-1)$ -multiindex β , by an easy computation, \mathcal{L}^β evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{z}^\beta}$. Therefore, we can just choose the same β^1, \dots, β^N as in Remark 5.3. ■

5.2 Type II spaces

In this section, we verify Hypothesis (I) for the orthogonal Grassmannians: $G_{II}(n, n)$. As shown in §2, we have a Zariski affine open affine coordinates chart as follows:

$$(I_{n \times n} \quad Z) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -z_{12} & 0 & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -z_{1n} & -z_{2n} & \cdots & 0 \end{pmatrix}$$

$z = (z_{12}, z_{13}, \dots, z_{(n-1)n})$ is the local coordinates for $\mathbb{C}^{\frac{n(n-1)}{2}} \cong \mathcal{A} \subset G_{II}(n, n)$; $\xi = (\xi_{12}, \dots, \xi_{(n-1)n})$ is the local coordinates for $\mathbb{C}^{\frac{n(n-1)}{2}} \cong \mathcal{A}^* \subset (G_{II}(n, n))^*$.

The canonical embedding is given by

$$(1, \dots, \text{pf}(Z_\sigma), \dots).$$

The defining function for the Segre family (in the product of such affine pieces) is given by

$$\rho(z, \xi) = 1 + \sum_{\substack{\sigma \in S_k, \\ 2 \leq k \leq n, 2|k}} \text{Pf}(Z_\sigma) \text{Pf}(\Xi_\sigma)$$

Write

$$r_Z = \left(\text{Pf}(Z_\sigma)_{\sigma \in S_k} \right)_{2 \leq k \leq n, 2|k}. \quad (69)$$

The map $F_j, j = 1, \dots, m$, can be represented as a matrix:

$$F_j = \begin{pmatrix} 0 & f_{j,12} & \cdots & f_{j,1n} \\ -f_{j,12} & 0 & \cdots & f_{j,2n} \\ \cdots & \cdots & \cdots & \cdots \\ -f_{j,1n} & \cdots & \cdots & 0 \end{pmatrix}, \quad j = 1, \dots, m$$

Let r_{F_j} be

$$r_{F_j} = \left(\text{pf}((F_j)_\sigma)_{\sigma \in S_k} \right)_{2 \leq k \leq n, 2|k}. \quad (70)$$

Under the notation of §2, it is easy to see $r_Z = (\psi_1, \dots, \psi_N)$, $r_{F_j} = (\psi_1(F_j), \dots, \psi_N(F_j))$.

We write \tilde{z} to be z with the last component $z_{(n-1)n}$ dropped. More precisely,

$$\tilde{z} = (z_{12}, \dots, z_{1n}, z_{23}, \dots, z_{2n}, \dots, z_{(n-2)(n-1)}, z_{(n-2)n}), \quad (71)$$

Recall z has $n' = n(n-1)/2$ independent variables. Thus \tilde{z} has $(n'-1)$ components. We define the \tilde{z} -rank and \tilde{z} -nondegeneracy as in Definition 3.1 using $\psi = r_{F_j}$ in (70) and \tilde{z} as in (71) with $m = n'$, respectively. We now prove the following:

Proposition 5.5. $r_{F_j}, j = 1, \dots, m$, is \tilde{z} -nondegenerate near 0. More precisely, $\text{rank}_{1+N-n'}(r_{F_j}, \tilde{z}) = N, j = 1, \dots, m$.

Proof of Proposition 5.5: Suppose not. Without loss of generality, we assume that

$$\text{rank}_{1+N-n'}(r_{F_1}, \tilde{z}) < N.$$

As a consequence of Theorem 3.11, there exist $c_{\sigma,k} \in \mathbb{C}, 4 \leq k \leq n, 2|k, \sigma \in S_k$, which are not all zero, such that

$$\sum_{4 \leq k \leq n, 2|k} \sum_{\sigma \in S_k} c_{\sigma,k} \text{pf}((F_1)_\sigma)(z_{12}, \dots, z_{(n-2)n}, 0) \equiv 0. \quad (72)$$

However, (72) cannot hold by the following lemma, which gives a contradiction:

Lemma 5.6. *Let*

$$H = \begin{pmatrix} 0 & h_{12} & \dots & h_{1n} \\ -h_{12} & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -h_{1n} & \dots & \dots & 0 \end{pmatrix}$$

be an anti-symmetric matrix-valued holomorphic function in a neighborhood U of 0 in $\tilde{Z} = (z_{12}, \dots, z_{(n-2)n}) \in \mathbb{C}^{n'-1}$ with $H(0) = 0$. Assume that H is of full rank at 0. Set r_H similar to the definition of r_F ,

$$r_H = \left(\text{pf}(H_\sigma)_{\sigma \in S_k} \right)_{2 \leq k \leq n, 2|k} \quad (73)$$

Assume that $a_{\sigma,k}, \sigma \in S_k, 4 \leq k \leq n$, are complex numbers such that

$$\sum_{4 \leq k \leq n, 2|k} \sum_{\sigma \in S_k} a_{\sigma,k} \text{pf}(H_\sigma)(z_{12}, \dots, z_{(n-2)n}) \equiv 0 \text{ for all } \tilde{Z} \in U. \quad (74)$$

Then

$$a_{\sigma,k} = 0$$

for all $\sigma \in S_k, 4 \leq k \leq n, 2|k$.

Proof of Lemma 5.6: Suppose not. We will prove the lemma by seeking a contradiction. Note that H has full rank at 0. Hence there exist $(n' - 1)$ components \hat{H} of H that forms a local biholomorphism from $\mathbb{C}^{n'-1}$ to $\mathbb{C}^{n'-1}$. We assume that these $(n' - 1)$ components \hat{H} are H with $h_{i_0 j_0}$ being dropped, where $i_0 < j_0$. Without loss of generality, we assume $i_0 = n - 1, j_0 = n$. By a local biholomorphic change of coordinates, we assume $\hat{H} = (z_{12}, \dots, z_{(n-2)n})$. We still write the missing component as $h_{(n-1)n}$. Now we assume $2(m+1), m \geq 1$, is the least number k such that $\{a_{\sigma,k}\}_{\sigma \in S_k}$ are not all zero. We then consider $\{a_{\sigma,2(m+1)}\}_{\sigma \in S_{2(m+1)}}$. We first claim that $a_{\sigma,2(m+1)} = 0$ for those $\sigma \in S_{2(m+1)}$ such that $\text{pf}(H_\sigma)$ involves $h_{(n-1)n}$. More precisely, if $\text{pf}(H_\sigma), \sigma \in S_{2(m+1)}$ involves $h_{(n-1)n}$, then $\sigma = \{i_1, \dots, i_{2m}, (n-1), n\}$ for some $1 \leq i_1 < \dots < i_{2m} \leq n - 2$. Suppose its coefficient is not zero. Then $\text{pf}(H_\sigma)$ will produce the monomial $z_{i_1 i_2} z_{i_3 i_4} \dots z_{i_{2m-3} i_{2m-2}} z_{i_{2m-1} (n-1)} z_{i_{2m} n}$. This term can only be cancelled by the terms

of form: $z_{i_{2m-1}(n-1)}h_{(n-1)n}Q$ or $z_{i_{2m}n}h_{(n-1)n}Q$. But neither of them can appear in any Pfaffian. This is a contradiction. Once we know there are no $h_{(n-1)n}$ involved, then the remaining Pfaffians are only in terms of $z_{12}, \dots, z_{(n-2)n}$. Their sum cannot be zero unless their coefficients are all zero. This is a contradiction. We thus establishes Lemma 5.6.

■

We thus get a contradiction to equation (72). This establishes Proposition 5.5.

■

Remark 5.7. Let F be any of F_1, \dots, F_m . By proposition 5.5, there exist multiindices $\tilde{\beta}^1, \dots, \tilde{\beta}^N$, with all $|\tilde{\beta}^j| \leq 1 + N - n'$, and

$$z^0 = \begin{pmatrix} 0 & z_{12}^0 & \dots & z_{1(n-1)}^0 & z_{1n}^0 \\ -z_{12}^0 & 0 & \dots & z_{2(n-1)}^0 & z_{2n}^0 \\ \dots & \dots & \dots & \dots & \dots \\ -z_{1(n-1)}^0 & -z_{2(n-1)}^0 & \dots & 0 & z_{(n-1)n}^0 \\ -z_{1n}^0 & -z_{2n}^0 & \dots & -z_{(n-1)n}^0 & 0 \end{pmatrix}, z_{(n-1)n}^0 \neq 0;$$

near 0, such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \bar{z}^{\tilde{\beta}^1}} & \dots & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \bar{z}^{\tilde{\beta}^1}} \\ \dots & \dots & \dots \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \bar{z}^{\tilde{\beta}^N}} & \dots & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \bar{z}^{\tilde{\beta}^N}} \end{vmatrix} (z^0) \neq 0. \quad (75)$$

We set

$$\xi^0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \xi_{(n-1)n}^0 \\ 0 & 0 & \dots & -\xi_{(n-1)n}^0 & 0 \end{pmatrix} \in \mathbb{C}^{n^2}, \xi_{(n-1)n}^0 = -\frac{1}{z_{(n-1)n}^0},$$

then it is easy to see that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write for each $1 \leq i < j \leq n, (i, j) \neq (n-1, n)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z, \xi)}{\frac{\partial \rho}{\partial z_{(n-1)n}}(z, \xi)} \frac{\partial}{\partial z_{(n-1)n}} \quad (76)$$

which are holomorphic tangent vectors along \mathcal{M} near (Z^0, ξ^0) . Here we note that $\frac{\partial \rho}{\partial z_{(n-1)n}}(z, \xi)$ is nonzero near (z^0, ξ^0) . For any $(n'-1)$ -multiindex $\beta = (\beta_{12}, \dots, \beta_{(n-2)n})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{12}^{\beta_{12}} \dots \mathcal{L}_{(n-2)n}^{\beta_{(n-2)n}}.$$

Now we define for any N collection of $(n' - 1)$ -multiindices $\{\beta^1, \dots, \beta^N\}$,

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix} (z, \xi). \quad (77)$$

Note that for any multiindex β, \mathcal{L}^β evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{z}^\beta}$. We thus again have

Theorem 5.8. *There exists multiindices $\{\beta^1, \dots, \beta^N\}$, such that*

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \neq 0,$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, \dots, 0)$.

5.3 Type III spaces

Let F be any of F_1, \dots, F_m . In this case, both Z and F are $n \times n$ symmetric matrices. We write

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{12} & z_{22} & \dots & z_{2n} \\ \dots & \dots & \dots & \dots \\ z_{1n} & z_{2n} & \dots & z_{nn} \end{pmatrix}, \quad z = (z_{11}, z_{12}, z_{13}, \dots, z_{nn}).$$

and similar notations are used for F .

Recall from (13) of ♣3 in §2:

$$r_z = \left(\psi_1^1, \dots, \psi_{N_1}^1, \psi_1^2, \dots, \psi_{N_2}^2, \dots, \psi_1^{n-1}, \dots, \psi_{N_{n-1}}^{n-1}, \psi^n \right), \quad (78)$$

where ψ_j^k is a homogeneous polynomial of degree k , $1 \leq j \leq N_k$. ψ^n is a homogeneous polynomial of degree n . Moreover, the components of r_z are linearly independent.

We write the number of components of r_z as $N = N_1 + \dots + N_n$, where we set $N_n = 1$. We will also sometimes write $\psi_{N_n}^n = \psi^n$.

We emphasize that for each fixed k , $\psi_1^k, \dots, \psi_{N_k}^k$ are linearly independent. Moreover, each ψ_j^k is a certain linear combination of the determinants of all $k \times k$ submatrices of Z . This will be crucial for our argument later.

We define r_F as the composition of r_z with the map F :

$$r_F = \left(\psi_1^1(F), \dots, \psi_{N_1}^1(F), \psi_1^2(F), \dots, \psi_{N_2}^2(F), \dots, \psi_1^{n-1}(F), \dots, \psi_{N_{n-1}}^{n-1}(F), \psi^n(F) \right), \quad (79)$$

In what follows, we write also $z_{ij} = z_{ji}$.

Lemma 5.9. Fix $1 \leq i, j < n$, and a monomial $P = z_{in}z_{nj}Q$ with Q also a monomial. Write $\tilde{P} = z_{ij}z_{nn}Q$. For a square submatrix of Z , its determinant either has none of terms of the form $\{P, \tilde{P}\}$; or have both of them with proportional coefficients of fixed ratio. More precisely, if $z_{ij} \nmid Q$, then the determinant of the submatrix always has monomials of the form $cP, -c\tilde{P}$ for some $c \in \mathbb{C}$ depending on the submatrix. (Hence its determinant has a term cP if and only if it has a term of the form $-c\tilde{P}$. If its determinant has none of them, it corresponds to $c = 0$). If $z_{ij} \mid Q$, then its determinant always has monomials $cP, -(c/2)\tilde{P}$ for some $c \in \mathbb{C}$ depending on the submatrix.

Proof of Lemma 5.9: This is a consequence of Laplace expansion of a determinant by complementary minors. ■

Similarly, one can prove in a similar way Lemmas 5.10-5.12. Here when we say the determinant always has $aP, b\tilde{P}$, we mean the determinant has a term of the form aP if and only if it has the term of the form $b\tilde{P}$.

Lemma 5.10. Fix $1 \leq j < n - 1$, and any monomial $P = z_{jn}z_{(n-1)(n-1)}Q$, and write $\tilde{P} = z_{j(n-1)}z_{(n-1)n}Q$. For every square submatrix of Z , if $z_{jn} \nmid Q$, then its determinant always has monomials $cP, -c\tilde{P}$ for some $c \in \mathbb{C}$. If $z_{jn} \mid Q$, then its determinant always has monomials $cP, -2c\tilde{P}$ for some $c \in \mathbb{C}$.

Lemma 5.11. Fix $1 \leq i < n - 1$, and any monomial $P = z_{i(n-1)}z_{ni}Q$, and write $\tilde{P} = z_{ii}z_{(n-1)n}Q$. For every square submatrix of Z , if $z_{(n-1)n} \nmid Q$, then its determinant always has monomials $cP, -c\tilde{P}$ for some $c \in \mathbb{C}$. If $z_{(n-1)n} \mid Q$, then its determinant always has monomials $cP, -(c/2)\tilde{P}$ for some $c \in \mathbb{C}$.

Lemma 5.12. Fix $1 \leq i < n - 1, 1 \leq j < n - 1, i \neq j$ and any monomial $P_1 = z_{i(n-1)}z_{nj}Q, P_2 = z_{in}z_{j(n-1)}Q$ and write $\tilde{P} = z_{ij}z_{(n-1)n}Q$. For every square submatrix of Z , if $z_{ij} \nmid Q, z_{(n-1)n} \nmid Q$ then its determinant always has terms $c_1P_1 + c_2P_2, -(c_1 + c_2)\tilde{P}$ for some $c_1, c_2 \in \mathbb{C}$. If $z_{ij} \nmid Q, z_{(n-1)n} \mid Q$, or $z_{ij} \mid Q, z_{(n-1)n} \nmid Q$, then its determinant always has terms $c_1P_1 + c_2P_2, -\frac{c_1+c_2}{2}\tilde{P}$ for some $c_1, c_2 \in \mathbb{C}$. If $z_{ij} \mid Q, z_{(n-1)n} \mid Q$, then its determinant always has terms $c_1P_1 + c_2P_2, -\frac{c_1+c_2}{4}\tilde{P}$.

As a consequence of Lemma 5.9, we have,

Lemma 5.13. Fix $1 \leq i, j < n$, and any monomial $P = z_{in}z_{nj}Q$ of degree k , and write $\tilde{P} = z_{ij}z_{nn}Q$. Then for each $\psi_l^k, 1 \leq l \leq N_k$, it either has none of $\{P, \tilde{P}\}$, or have both of them with proportional coefficients with fixed ratio. More precisely, if $z_{ij} \nmid Q$, then ψ_l^k always has monomials $\lambda P, -\lambda\tilde{P}$ for some $\lambda \in \mathbb{C}$. If $z_{ij} \mid Q$, then ψ_l^k always has monomials $\lambda P, -(\lambda/2)\tilde{P}$ for some $\lambda \in \mathbb{C}$.

Proof of Lemma 5.13: This is due to the fact that ψ_j^k is a linear combination of the determinants of all submatrices of Z of order k . This is then a corollary of Lemma 5.9. ■

Remark 5.14. One easily sees that, for each lemma from 5.10-5.12, there is a similar consequence as in Lemma 5.13. For instance, as a corollary of Lemma 5.12, we have

Lemma 5.15. Fix $1 \leq i < n-1, 1 \leq j < n-1, i \neq j$ and any monomial $P_1 = z_{i(n-1)}z_{nj}Q, P_2 = z_{in}z_{j(n-1)}Q$ and write $\tilde{P} = z_{ij}z_{(n-1)n}Q$. For each $\psi_l^k, 1 \leq l \leq N_k$, if $z_{ij} \nmid Q, z_{(n-1)n} \nmid Q$ then ψ_l^k always has terms $\lambda_1 P_1 + \lambda_2 P_2, -(\lambda_1 + \lambda_2)\tilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. If $z_{ij} \nmid Q, z_{(n-1)n} \mid Q$, or $z_{ij} \mid Q, z_{(n-1)n} \nmid Q$, then it always has terms $\lambda_1 P_1 + \lambda_2 P_2, -\frac{\lambda_1 + \lambda_2}{2}\tilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. If $z_{ij} \mid Q, z_{(n-1)n} \mid Q$, then its determinant always has terms $\lambda_1 P_1 + \lambda_2 P_2, -\frac{\lambda_1 + \lambda_2}{4}\tilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$.

We write \tilde{z} for z with the last components z_{nn} being dropped. More precisely,

$$\tilde{z} = (z_{11}, \dots, z_{1n}, z_{22}, \dots, z_{2n}, \dots, z_{(n-1)(n-1)}, z_{(n-1)n}). \quad (80)$$

Recall z has $n' = n(n+1)/2$ independent variables. Thus \tilde{z} has $(n'-1)$ components. We define \tilde{z} -rank and \tilde{z} -nondegeneracy in the same way as in Definition 1.6, using r_F in (79) and \tilde{z} in (80) with $m = n'$. We now need to prove the following:

Proposition 5.16. r_F is \tilde{z} -nondegenerate at 0. More precisely, $\text{rank}_{1+N-n'}(r_F, \tilde{z}) = N$.

Proof of Proposition 5.16: Suppose not. Then one easily verifies that the hypothesis of Theorem 3.11 is satisfied. As a consequence of Theorem 3.11, There exist $c_j^k \in \mathbb{C}, 2 \leq k \leq n, 1 \leq j \leq N_k$, which are not all zero such that

$$\sum_{k=2}^n \sum_{j=1}^{N_k} c_j^k \psi_j^k(F(z_{11}, \dots, z_{(n-1)n}, 0)) \equiv 0. \quad (81)$$

Here recall we write $N_n = 1, \psi_{N_n}^n = \psi^n$.

Then we just need to show it cannot hold by the following lemma.

Lemma 5.17. Let

$$H = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{12} & h_{22} & \dots & h_{2n} \\ \dots & \dots & \dots & \dots \\ h_{1n} & \dots & \dots & h_{nn} \end{pmatrix}$$

be a symmetric matrix-valued holomorphic function in $\tilde{z} = (z_{11}, \dots, z_{1n}, z_{22}, \dots, z_{2n}, \dots, z_{(n-1)(n-1)}, z_{(n-1)n}) \in \mathbb{C}^{n'-1}$ with $H(0) = 0$. Assume that H is full rank at 0. Set r_H in a similar way as in (36):

$$r_H = \left(\psi_1^1(H), \dots, \psi_{N_1}^1(H), \psi_1^2(H), \dots, \psi_{N_2}^2(H), \dots, \psi_1^{n-1}(H), \dots, \psi_{N_{n-1}}^{n-1}(H), \psi^n(H) \right)$$

Again we write $N_n = 1, \psi^n = \psi_{N_n}^n$. Assume that $a_j^k, 2 \leq k \leq n, 1 \leq j \leq n$ are complex numbers such that

$$\sum_{k=2}^n \sum_{j=1}^{N_k} a_j^k \psi_j^k(H(\tilde{z})) \equiv 0 \quad \text{for } \tilde{z} \in U. \quad (82)$$

Then

$$a_j^k = 0$$

for all $2 \leq k \leq n, 1 \leq j \leq N_k$.

Proof of Lemma 5.17: Suppose not. We will prove by seeking a contradiction. Notice that H has full rank at 0. Hence there exist $(n' - 1)$ components \hat{H} of H that gives a local biholomorphism from $\mathbb{C}^{n'-1}$ to $\mathbb{C}^{n'-1}$. We assume these $(n' - 1)$ components \hat{H} are H with $h_{i_0 j_0}$ being dropped, where $i_0 \leq j_0$. Then we split our argument into two parts in terms of $i_0 = j_0$ or $i_0 < j_0$.

Case I: Assume that $i_0 = j_0$. Without loss of generality, we assume $i_0 = j_0 = n$. By a local biholomorphic change of coordinates, we assume $\hat{H} = (z_{11}, \dots, z_{n(n-1)})$. We still write the last component as h_{nn} . Now we assume $m \geq 2$ is the least number k such that $\{a_1^k, \dots, a_{N_k}^k\}$ are not all zero. For any holomorphic g , we define $T_l(g)$ to be the homogeneous part of degree l in the Taylor expansion of g at 0. Now the assumption in (82) yields:

$$T_m \left(\sum_{j=1}^{N_m} a_j^m \psi_j^m(H(\tilde{z})) \right) \equiv 0. \quad (83)$$

We first compute

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(H) = \sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn})$$

formally. Namely, we regard h_{nn} as a formal variable and only conduct formal cancelations. We write formally

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn}) = P_1 + h_{nn} P_2. \quad (84)$$

Here $P_1 = P_1(z_{11}, \dots, z_{(n-1)n})$ is a homogeneous polynomial of degree m . $P_2 = P_2(z_{11}, \dots, z_{(n-1)n})$ is a homogeneous polynomial of degree of degree $m - 1$. We claim $P_2 \neq 0$. Otherwise,

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn}) = P_1$$

formally, this implies $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn})$ does not depend on h_{nn} formally. Then we can replace h_{nn} by z_{nn} . That is,

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, z_{nn}) = \sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn}(\tilde{z})) = P_1. \quad (85)$$

By (83), we have (85) is identically zero. This is a contradiction to the fact that $\{\psi_1^m, \dots, \psi_{N_m}^m\}$ is linearly independent.

Now since $P_2 \neq 0$, thus by (84), $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn})$ has a monomial of form $\mu \tilde{P} = \mu z_{ij} h_{nn} Q$ of degree m for some $1 \leq i, j < n, \mu \neq 0$ and some monomial Q . By Lemma 5.13, we get that $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, \dots, z_{(n-1)n}, h_{nn})$ has also the term $-\mu P$ or $-2\mu P$, where $P = z_{in} z_{nj} Q$. This is a contradiction to (83). Indeed, P can be only cancelled by the terms of the forms: $z_{in} h_{nn} \tilde{Q}$ or $z_{nj} h_{nn} \tilde{Q}$, where \tilde{Q} is of degree $m-2$. But they cannot appear in determinant of any submatrix of H as z_{in} (or z_{nj}) can not appear with h_{nn} .

Case II: Assume that $i_0 \neq j_0$. Without loss of generality, we assume $i_0 = (n-1), j_0 = n$. Then $\hat{H} = (h_{11}, \dots, h_{(n-1)(n-1)}, h_{nn})$ is a local biholomorphism. By a local biholomorphic change of coordinates, we assume $\hat{H} = (z_{11}, \dots, z_{(n-1)n})$. We will still write the remaining component as $h_{(n-1)n} = h_{n(n-1)}$. Note that the fact we are using only is that $(z_{11}, \dots, z_{(n-1)n})$ are independent variables. Hence, to make our notation easier, we will write

$$\hat{H} = (z_{11}, \dots, z_{(n-1)n}) = (w_{11}, \dots, w_{1n}, w_{22}, \dots, w_{2n}, \dots, w_{(n-1)(n-1)}, w_{nn})$$

such that they have the same indices as h 's in \hat{H} . Now we assume m is the least number k such that $\{a_1^k, \dots, a_{N_k}^k\}$ are not all zero. Then again assumption (82) yields that

$$T_m \left(\sum_{j=1}^{N_m} a_j^m \psi_j^m(H(\tilde{Z})) \right) \equiv 0. \quad (86)$$

Again we formally compute that

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(w_{11}, \dots, h_{(n-1)n}, w_{nn}) = Q_1 + h_{(n-1)n} Q_2. \quad (87)$$

Here $Q_1 = Q_1(w_{11}, \dots, w_{(n-1)(n-1)}, w_{nn})$ is a homogeneous polynomial of degree m . Similarly, we can show that $Q_2 \neq 0$. We claim that (87) does not have a monomial of the form $h_{(n-1)n} h_{(n-1)n} Q$. Otherwise, by Remark 5.14 (Lemma 5.9), we get that (87) has also a monomial of degree m of the form: $w_{(n-1)(n-1)} w_{nn} Q$. But note that it can be cancelled only by $h_{(n-1)n} h_{(n-1)n} Q$. Then $h_{(n-1)n}$ will have a linear term $w_{(n-1)(n-1)}$. But then $h_{(n-1)n} h_{(n-1)n} Q$ will produce the term $w_{(n-1)(n-1)} w_{(n-1)(n-1)} Q$. This cannot be canceled out by any other terms.

Now since $Q_2 \neq 0$, (87) has a monomial of the form $w_{ij} h_{(n-1)n} Q$, where Q is another monomial in w 's. Here $1 \leq i, j \leq n$. Moreover, $(i, j) \neq (n-1, n-1), (n-1, n), (n, n-1)$ or (n, n) . We first assume $1 \leq i, j < n-1, i \neq j$. Then by Lemma 5.15, (87) has either P_1 or P_2 , where $P_1 = w_{i(n-1)} w_{nj} Q, P_2 = w_{in} w_{j(n-1)} Q$. Note P_1, P_2 can only be cancelled by the terms $w_{i(n-1)} h_{(n-1)n} Q, w_{nj} h_{(n-1)n} Q, w_{in} h_{(n-1)n} Q, w_{j(n-1)} h_{(n-1)n} Q$. So one of them will appear in (87). Whichever case it is, by Remark 5.14 (Lemma 5.9, 5.10), (87) will have either $P = w_{ln} w_{(n-1)(n-1)} Q$, or $\hat{P} = w_{l(n-1)} w_{nn} Q$ for some $1 \leq l < n$. We assume for instance,

(87) has the monomial P . Then it also has the monomial $\tilde{P} = w_{l(n-1)}h_{(n-1)n}Q$ by Remark 5.14(Lemma 5.10). Note P can be only cancelled by $w_{ln}h_{n(n-1)}Q$. Hence $h_{n(n-1)}$ has a linear $w_{(n-1)(n-1)}$ term. Then \tilde{P} will have the monomial $w_{l(n-1)}w_{(n-1)(n-1)}Q$, which cannot be cancelled by any other terms. This is a contradiction. The other cases can be proved similarly.

■

This establishes Proposition 5.16.

■

Remark 5.18. By Proposition 5.16, there exist multiindices $\tilde{\beta}^1, \dots, \tilde{\beta}^N$ with $|\tilde{\beta}^j| \leq 1 + N - pq$, and there exist

$$z^0 = \begin{pmatrix} z_{11}^0 & \dots & z_{1n}^0 \\ \dots & \dots & \dots \\ z_{1n}^0 & \dots & z_{nn}^0 \end{pmatrix}, z_{nn}^0 \neq 0,$$

near 0 such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{Z}^{\tilde{\beta}^1}} & \dots & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \tilde{Z}^{\tilde{\beta}^1}} \\ \dots & \dots & \dots \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \tilde{Z}^{\tilde{\beta}^N}} & \dots & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \tilde{Z}^{\tilde{\beta}^N}} \end{vmatrix} (z^0) \neq 0. \quad (88)$$

Here we simply write $r_F = (\psi_1(F), \dots, \psi_N(F))$.

We then set

$$\xi^0 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \xi_{nn}^0 \end{pmatrix} \in \mathbb{C}^{n^2}, \xi_{nn}^0 = -\frac{1}{z_{nn}^0}.$$

It is easy to verify that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write for each $1 \leq i \leq j \leq n, (i, j) \neq (n, n)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z, \xi)}{\frac{\partial \rho}{\partial z_{nn}}(z, \xi)} \frac{\partial}{\partial z_{nn}}, \quad (89)$$

which are holomorphic tangent vector fields along \mathcal{M} near (z^0, ξ^0) . Here we note that $\frac{\partial \rho}{\partial z_{nn}}(z, \xi)$ is nonzero near (z^0, ξ^0) . For any $(n' - 1)$ -multiindex $\beta = (\beta_{11}, \dots, \beta_{(n-1)n})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{11}^{\beta_{11}} \dots \mathcal{L}_{(n-1)n}^{\beta_{(n-1)n}}.$$

Now we define for any N collection of $(n' - 1)$ -multiindices $\{\beta^1, \dots, \beta^N\}$,

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix} (z, \xi). \quad (90)$$

Note \mathcal{L}^β evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial Z^\beta}$. We have

Theorem 5.19. *There exists multiindices $\{\beta^1, \dots, \beta^N\}$, such that $\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \neq 0$ for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, 0, \dots, 0)$.*

5.4 Spaces of Type IV

In this subsection, we consider the hyperquadric case. This case is more subtle because the tangent vector fields of its Segre variety are more complicated.

Recall that Q^n is defined by

$$\left\{ (z_0, \dots, z_{n+1}) \in \mathbb{P}^{n+1} : \sum_{i=1}^n z_i^2 - 2z_0 z_{n+1} = 0 \right\},$$

where $[z_0, \dots, z_{n+1}]$ is the homogeneous coordinates of \mathbb{P}^{n+1} . A natural embedding $\mathbb{C}^n \rightarrow \mathcal{A} \subset Q^n$ is given by

$$z := (z_1, \dots, z_n) \mapsto [1, \psi_1(z), \dots, \psi_{n+1}(z)] = [1, z_1, \dots, z_n, \frac{1}{2} \sum_{i=1}^n z_i^2].$$

The defining function of the Segre family over $\mathcal{A} \times \mathcal{A}$ is $\rho(z, \xi) = 1 + r_z \cdot r_\xi$, where

$$r_z = (z_1, \dots, z_n, \frac{1}{2} \sum_{i=1}^n z_i^2), r_\xi = (\xi_1, \dots, \xi_n, \frac{1}{2} \sum_{i=1}^n \xi_i^2). \quad (91)$$

Let F be any of F_1, \dots, F_m . We define

$$F = (f_1, \dots, f_n), \quad r_F = (f_1, \dots, f_n, \frac{1}{2} \sum_{i=1}^n f_i^2). \quad (92)$$

Notice that

$$r_Z = (\psi_1(z), \dots, \psi_{n+1}(z)), r_F = (\psi_1(F), \dots, \psi_{n+1}(F))$$

We will need the following lemma:

Lemma 5.20. *For each fixed μ_1, \dots, μ_{n-1} with $(\sum_{i=1}^{n-1} \mu_i^2) + 1 = 0$ and each fixed (z_1, \dots, z_{n-1}) with $(\sum_{i=1}^{n-1} \mu_i z_i) + z_n \neq 0$, we can find (ξ_1, \dots, ξ_n) such that*

$$1 + z_1 \xi_1 + \dots + z_n \xi_n = 0; \quad \sum_{i=1}^n (\xi_i)^2 = 0, \quad \xi_j = \mu_j \xi_n, 1 \leq j \leq n-1, \quad \xi_n \neq 0. \quad (93)$$

Proof of Lemma 5.20: We just need to set

$$\xi_n = \frac{-1}{(\sum_{i=1}^{n-1} \mu_i z_i) + z_n}, \quad \xi_j = \mu_j \xi_n, 1 \leq j \leq n-1.$$

Then it is easy to verify that (93) is satisfied. ■

Recall that in the Type I case, the tangent vector fields in \mathbb{C}^{pq} , that we applied there, are $\frac{\partial}{\partial \bar{z}^\alpha}$. They are tangent vector fields of the particular hyperplane $\{z_{pq} = 0\}$. We can indeed formulate this result in a abstract way and extend it to a more general setting. For instance, it can be generalized to the any complex hyperplane case. We will briefly discuss this in the following:

First fix μ_1, \dots, μ_{n-1} with $(\sum_{i=1}^{n-1} \mu_i^2) + 1 = 0$. Take the complex hyperplane $\mathbb{H} : z_n + \sum_{i=1}^{n-1} \mu_i z_i = 0$ in $(z_1, \dots, z_n) \in \mathbb{C}^n$.

Write

$$L_1 = \frac{\partial}{\partial z_1} - \mu_1 \frac{\partial}{\partial z_n}, \dots, L_{n-1} = \frac{\partial}{\partial z_{n-1}} - \mu_{n-1} \frac{\partial}{\partial z_n}.$$

Then $\{L_i\}_{i=1}^{n-1}$ forms a basis of the tangent vector fields of \mathbb{H} . For any multiindex $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, we write $L^\alpha = L_1^{\alpha_1} \dots L_{n-1}^{\alpha_{n-1}}$. We define L -rank and L -nondegeneracy as in Definition 3.1 using r_F in (92) and L^α instead of \tilde{z}^α with $m = n$. We write the k th L -rank defined in this setting as $\text{rank}_k(r_F, L)$. We now need to prove the following

Proposition 5.21. *F is L -nondegenerate near 0. More precisely, $\text{rank}_2(r_F, L) = n + 1$.*

Proof of Proposition 5.21: Suppose not. By applying the same argument as in Section 3 and a linear change of coordinates, we are able to obtain a modified version of Theorem 3.11. That is, we can arrive the following:

Lemma 5.22. *There exist $n+1$ holomorphic functions $g_1(w), \dots, g_{n+1}(w)$ near 0 on the w -plane with $\{g_1(0), \dots, g_{n+1}(0)\}$ not all zero such that the following holds for all $z \in U$.*

$$\sum_{i=1}^{n+1} g_i(z_n + \mu_1 z_1 + \dots + \mu_{n-1} z_{n-1}) \psi_i(F(z)) \equiv 0. \quad (94)$$

Then one shows with a similar argument as in Section 3, by the fact that F has full rank at 0, that $g_1(0) = 0, \dots, g_n(0) = 0$. Hence we obtain,

Lemma 5.23. *There exists a non-zero constant $c \in \mathbb{C}$ such that*

$$c \psi_{n+1}(F(z)) = \frac{c}{2} \sum_{i=1}^n f_i^2(z) \equiv 0, \quad (95)$$

for all $z \in U$ when restricted on $z_n + \sum_{i=1}^{n-1} \mu_i z_i = 0$.

We then just need to show that (95) cannot hold by applying the following Lemma and a linear change of coordinates.

Lemma 5.24. *Let $H = (h_1, \dots, h_n)$ be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ with $H(0) = 0$. Assume that H is full rank at 0. Assume that a is a complex number such that,*

$$a \sum_{i=1}^n h_i^2(\tilde{z}) \equiv 0, \quad (96)$$

Then $a = 0$.

Proof of Lemma 5.34: Suppose not. We will prove by seeking a contradiction. Note H is full rank at 0. We assume, without loss of generality, that (h_1, \dots, h_{n-1}) gives a local biholomorphic map near 0 from \mathbb{C}^{n-1} to \mathbb{C}^{n-1} . By a local biholomorphic change of coordinates, we assume $(h_1, \dots, h_{n-1}) = (z_1, \dots, z_{n-1})$, and still write the last component as h_n . Then equation (96) is reduced to

$$a(z_1^2 + \dots + z_{n-1}^2 + h_n^2) = 0.$$

To cancel the z_1^2, z_2^2 terms, it yields that h_n has linear z_1, z_2 terms. But then h_n^2 will have a $z_1 z_2$ term, which cannot be canceled out by any other terms. This is a contradiction.

■

This establishes Proposition 5.21.

■

Remark 5.25. *By Proposition 5.21, there exist multiindices $\tilde{\beta}^1, \dots, \tilde{\beta}^{n+1}$ with $|\tilde{\beta}^j| \leq 2$ and*

$$z^0 = (z_1^0, \dots, z_n^0) \text{ with } \sum_{i=1}^{n-1} \mu_i z_i^0 + z_n^0 \neq 0$$

such that

$$\begin{vmatrix} L^{\tilde{\beta}^1}(\psi_1(F)) & \dots & L^{\tilde{\beta}^1}(\psi_{n+1}(F)) \\ \dots & \dots & \dots \\ L^{\tilde{\beta}^{n+1}}(\psi_1(F)) & \dots & L^{\tilde{\beta}^{n+1}}(\psi_{n+1}(F)) \end{vmatrix} (z^0) \neq 0. \quad (97)$$

We then choose $\xi^0 = (\xi_1^0, \dots, \xi_n^0)$ as in Lemma 5.20. That is

$$1 + z_1^0 \xi_1^0 + \dots + z_n^0 \xi_n^0 = 0; \quad \sum_{i=1}^n (\xi_i^0)^2 = 0, \quad \xi_j^0 = \mu_j \xi_n^0, 1 \leq j \leq n-1, \quad \xi_n^0 \neq 0.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M}$.

We now define

$$\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z, \xi)}{\frac{\partial \rho}{\partial z_n}(z, \xi)} \frac{\partial}{\partial z_n}, 1 \leq i \leq n-1 \quad (98)$$

for $(z, \xi) \in \mathcal{M}$ near (z^0, ξ^0) . They are tangent vector fields along \mathcal{M} . Moreover, $\frac{\partial \rho}{\partial z_n}(z, \xi)$ is nonzero near (z^0, ξ^0) .

We define for any multiindex $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1} \dots \mathcal{L}_{n-1}^{\alpha_{n-1}}$. Then we define, for any $(n+1)$ collection of $(n-1)$ -multiindices $\{\beta^1, \dots, \beta^N\}$,

$$\Lambda(\beta^1, \dots, \beta^{n+1})(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_{n+1}(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^{n+1}}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^{n+1}}(\psi_{n+1}(F)) \end{vmatrix} (z, \xi). \quad (99)$$

By the fact that $\sum_{i=1}^n (\xi_i^0)^2 = 0$, one can check that, for any multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, $\mathcal{L}^\alpha = L^\alpha$ when evaluated at (z^0, ξ^0) . Then we get the following:

Theorem 5.26. *There exist multiindices $\{\beta^1, \dots, \beta^N\}$ such that*

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \neq 0,$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) , where $\beta^1 = (0, 0, \dots, 0)$

5.5 The exceptional class: \mathbf{M}_{27}

In this setting, we use the coordinates

$$z = (x_1, x_2, x_3, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7) \in \mathbb{C}^{27}.$$

The defining function of the Segre family described in (17) is :

$$\rho(z, \xi) = 1 + r_z \cdot r_\xi = 1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi), \quad \text{where } N = 55, \text{ and}$$

$$r_z = (x_1, x_2, x_3, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7, A, B, C, D_0, \dots, D_7, E_0, \dots, E_7, F_0, \dots, F_7, G). \quad (100)$$

Here A, B, C, D_i, E_i, F_i are homogeneous quadratic polynomials in z and G is a homogeneous cubic polynomial in z :

$$A = x_2 x_3 - \sum_{i=0}^7 w_i^2, B = x_1 x_3 - \sum_{i=0}^7 t_i^2, C = x_1 x_2 - \sum_{i=0}^7 y_i^2. \quad (101)$$

For the expressions for D_i, E_i, F_i, G , see Appendix 2. Let F be any of F_1, \dots, F_m . We write

$$F = (\phi_1, \phi_2, \phi_3, f_{10}, \dots, f_{17}, f_{20}, \dots, f_{27}, f_{30}, \dots, h_{37}).$$

Also define r_F to be the composition of r_z with F :

$$r_F = r_z \circ F = (\phi_1, \phi_2, \phi_3, f_{10}, \dots, f_{17}, f_{20}, \dots, f_{27}, f_{30}, \dots, f_{37}, A(F), B(F), C(F), \dots, G(F)), \quad (102)$$

Notice that r_F has 55 components. We will also write

$$r_F = (\psi_1(F), \dots, \psi_{55}(F)).$$

We write \tilde{z} for z with x_3 being dropped. Namely,

$$\tilde{z} = (x_1, x_2, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7). \quad (103)$$

We define the \tilde{z} -rank and ψ -nondegeneracy as in Definition 3.1 using r_F in (102) and \tilde{z} in (103) with $m = 27$.

Proposition 5.27. *F is \tilde{z} -nondegenerate near 0. More precisely, $\text{rank}_{29}(F, \tilde{z}) = 55$.*

Proof of Proposition 5.27: Suppose not. As a consequence of Theorem 3.11, there exist $c_1, \dots, c_{28} \in \mathbb{C}$ that are not all zero, such that

$$c_1 A(F(x_1, x_2, 0, y_0, \dots, w_7)) + \dots + c_{28} G(F(x_1, x_2, 0, y_0, \dots, w_7)) \equiv 0. \quad (104)$$

We will show that (104) cannot hold by the following lemma:

Lemma 5.28. *Let $H = (\psi_1, \psi_2, \psi_3, h_{10}, \dots, h_{17}, h_{20}, \dots, h_{27}, h_{30}, \dots, h_{37})$ be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (x_1, x_2, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7) \in \mathbb{C}^{26}$ with $H(0) = 0$. Assume that H has full rank at 0. Assume that a_1, \dots, a_{28} are complex numbers such that,*

$$a_1 A(H(\tilde{z})) + \dots + a_{28} G(H(\tilde{z})) = 0 \text{ for all } \tilde{z} \in U. \quad (105)$$

Then $a_i = 0$ for all $1 \leq i \leq 28$.

Proof of Lemma 5.28: Suppose not. Notice that H has full rank at 0. Hence there exist 26 components \hat{H} of H that give a local biholomorphism from \mathbb{C}^{26} to \mathbb{C}^{26} . We assume these 26 components \hat{H} are the H with η dropped, where $\eta \in \{\psi_1, \psi_2, \psi_3, h_{10}, \dots, h_{17}, h_{20}, \dots, h_{27}, h_{30}, \dots, h_{37}\}$. By a local biholomorphic change of coordinates, we assume

$$\hat{H} = (x_1, x_2, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7).$$

We still write the remaining components as η .

Case I: If $\eta \in \{\psi_1, \psi_2, \psi_3\}$, without loss of generality, we can assume $\eta = \psi_3$. We first claim that the coefficients of A, B , i.e., a_1, a_2 are zero. This is due to the fact that $t_i^2, w_i^2, 0 \leq i \leq 7$ can only be canceled by $t_i \psi_3, w_i \psi_3$, which do not appear in the expressions of $A(H), \dots, G(H)$. We then claim the coefficients of C are zero, for $x_1 x_2$ can not be canceled. Then the coefficients of all D 's have to be zero, for each $t_i w_j$ is unique and can not be canceled. Then it follows trivially that all other coefficients are zero.

Case II: If $\eta \in \{h_{10}, \dots, h_{17}, h_{20}, \dots, h_{27}, h_{30}, \dots, h_{37}\}$, without loss of generality, we assume $\eta = h_{37}$. Notice that the only fact we are using about \hat{H} is that its components are independent variables. For simplicity of notation, we will write

$$\hat{H} = (x_1, x_2, x_3, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_6).$$

We first claim that the coefficient of A is zero. This is due to the fact that x_2x_3 cannot be canceled. We also claim that the coefficient of B is zero. Suppose not. Notice that t_i^2 can only be canceled by $t_i h_{37}$. Then the coefficient of each D_i is non zero for $0 \leq i \leq 7$. Moreover, x_1x_3 can only be canceled by $x_1 h_{37}$. This implies h_{37} has a linear x_3 -term. Then, in particular, the $t_7 h_{37}$ term in D_0 will produce a $t_7 x_3$ term. It cannot be canceled by any other terms. This is a contradiction. Similarly, one can show that the coefficient of C is zero. Then we claim the coefficient of D_0 is zero. Otherwise, to cancel the $x_3 y_0$ term, h_{37} needs have a linear x_3 term. Then the term $t_7 h_{37}$ in D_0 will produce a $t_7 x_3$ term, which cannot be canceled by any other term. By the same argument, one can show that the coefficients of all $D_i, 0 \leq i \leq 7$, are zero. Similarly, we can obtain the coefficients of all $E_i, 0 \leq i \leq 7$, are zero. Then we claim the coefficients of all F 's have to be zero. This is because each $y_i t_j$ is unique. It can not be canceled out. Finally we get the coefficient of G to be zero.

■

This establishes Proposition 5.27.

■

Remark 5.29. By Proposition 5.27, there exist multiindices $\tilde{\beta}^1, \dots, \tilde{\beta}^{55}$ with $|\tilde{\beta}^j| \leq 29$, and there exist

$$z^0 = (x_1^0, x_2^0, x_3^0, y_0^0, \dots, y_7^0, t_0^0, \dots, t_7^0, w_0^0, \dots, w_7^0), \quad x_3^0 \neq 0,$$

such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{z}^{\beta^1}} & \dots & \frac{\partial^{|\beta^1|}(\psi_{55}(F))}{\partial \tilde{z}^{\beta^1}} \\ \dots & \dots & \dots \\ \frac{\partial^{|\beta^{55}|}(\psi_1(F))}{\partial \tilde{z}^{\beta^{55}}} & \dots & \frac{\partial^{|\beta^{55}|}(\psi_{55}(F))}{\partial \tilde{z}^{\beta^{55}}} \end{vmatrix} (z^0) \neq 0.$$

Then we set

$$\xi^0 = (0, 0, \xi_3^0, 0, \dots, 0, \dots, 0, 0, \dots, 0), \quad \xi_3^0 = -\frac{1}{x_3^0}.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write

$$\begin{aligned} \mathcal{L}_i &= \frac{\partial}{\partial x_i} - \frac{\frac{\partial \rho}{\partial x_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, \quad 1 \leq i \leq 2; \\ \mathcal{L}_{3+i} &= \frac{\partial}{\partial y_i} - \frac{\frac{\partial \rho}{\partial y_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, \quad 0 \leq i \leq 7; \\ \mathcal{L}_{11+i} &= \frac{\partial}{\partial t_i} - \frac{\frac{\partial \rho}{\partial t_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, \quad 0 \leq i \leq 7; \end{aligned}$$

$$\mathcal{L}_{19+i} = \frac{\partial}{\partial w_i} - \frac{\frac{\partial \rho}{\partial w_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \leq i \leq 7.$$

For any 26-multiindex $\beta = (\beta_1, \dots, \beta_{26})$, we write $\mathcal{L}^\beta = \mathcal{L}_1^{\beta_1} \dots \mathcal{L}_{26}^{\beta_{26}}$. Now we define for any 55 collection of 26-multiindices $\{\beta^1, \dots, \beta^{55}\}$,

$$\Lambda(\beta^1, \dots, \beta^{55})(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_{55}(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^{55}}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^{55}}(\psi_{55}(F)) \end{vmatrix} (z, \xi). \quad (106)$$

Note that for any multiindex, \mathcal{L}^β evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{Z}^\beta}$. We have,

Theorem 5.30. *There exists multiindices $\{\beta^1, \dots, \beta^{55}\}$, such that*

$$\Lambda(\beta^1, \dots, \beta^{55})(z, \xi) \neq 0,$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, \dots, 0)$.

5.6 The exceptional class: \mathbf{M}_{16}

This case is very similar to the hyperquadric setting. In this case, we write the coordinates of \mathbb{C}^{16} as

$$z := (x_0, \dots, x_7, y_0, \dots, y_7).$$

The defining function of the Segre family as described in (16) is

$$\rho(z, \xi) = 1 + r_z \cdot r_\xi = 1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi), \quad \text{where } N = 26, \text{ and}$$

$$r_Z = (x_0, \dots, x_7, y_0, \dots, y_7, A_0, \dots, A_7, B_0, B_1). \quad (107)$$

Here $A_i, 0 \leq i \leq 7, B_0, B_1$ are homogeneous quadratic polynomials in z . For instance,

$$B_0 = \sum_{i=0}^7 x_i^2, B_1 = \sum_{i=0}^7 y_i^2.$$

For the expressions for A_i , see Appendix 2.

Let F be any of F_1, \dots, F_m . We write

$$F = (f_0, \dots, f_7, \tilde{f}_0, \dots, \tilde{f}_7).$$

And define r_F as the composition of r_z with F :

$$r_F = r_z \circ F = (f_0, \dots, f_7, \tilde{f}_0, \dots, \tilde{f}_7, A_0(F), \dots, A_7(F), B_0(F), B_1(F)), \quad (108)$$

Notice that r_F has 26 components.

We will need the following lemma:

Lemma 5.31. *For each fixed μ_0, \dots, μ_6 with $(\sum_{i=0}^6 \mu_i^2) + 1 = 0$ and fixed (y_0, \dots, y_7) with $(\sum_{i=0}^6 \mu_i y_i) + y_7 \neq 0$, we can always find (ξ_0, \dots, ξ_7) such that*

$$1 + y_0 \xi_0 + \dots + y_7 \xi_7 = 0; \quad \sum_{i=0}^7 (\xi_i)^2 = 0, \quad \xi_j = \mu_j \xi_7, 0 \leq j \leq 6, \quad \xi_7 \neq 0.$$

Proof of Lemma 5.31: The proof is similar to that as in the hyperquadric case. ■

Take the complex hyperplane $\mathbb{H} : y_7 + \sum_{j=0}^6 \mu_j y_j = 0$ in $(x_0, \dots, x_7, y_0, \dots, y_7) \in \mathbb{C}^{16}$. Write

$$L_0 = \frac{\partial}{\partial x_0}, \dots, L_7 = \frac{\partial}{\partial x_7}.$$

$$L_8 = \frac{\partial}{\partial y_0} - \mu_1 \frac{\partial}{\partial y_7}, \dots, L_{14} = \frac{\partial}{\partial y_6} - \mu_6 \frac{\partial}{\partial y_7}.$$

Then $\{L_i\}_{i=0}^{14}$ forms a basis of the tangent vector fields of \mathbb{H} . For any multiindex $\alpha = (\alpha_0, \dots, \alpha_{14})$, we write $L^\alpha = L_0^{\alpha_0} \dots L_{14}^{\alpha_{14}}$. We define L -rank and L -nondegeneracy as in Definition 3.1 using r_F in (108) and L^α instead of \tilde{z}^α . We write the k th L -rank defined in this setting as $\text{rank}_k(r_F, L)$. We now need to prove the following:

Proposition 5.32. *F is L -nondegenerate near 0. More precisely, $\text{rank}_{11}(r_F, L) = 26$.*

Proof of Proposition 5.32: Suppose not. As in the hyperquadric case, by a modified version of Theorem 3.11, we have that, there exist 26 holomorphic functions $g_0(w), \dots, g_{25}(w)$ near 0 on the w -plane with $\{g_0(0), \dots, g_{25}(0)\}$ not all zero such that the following holds for all $z \in U$.

$$\sum_{i=0}^{25} g_i(y_7 + \mu_0 y_0 + \dots + \mu_6 y_6) \psi_i(F(z)) \equiv 0. \quad (109)$$

Then one show similarly as before, by the fact that F is full rank at 0, that $g_0(0) = 0, \dots, g_{15}(0) = 0$. Hence we obtain,

Lemma 5.33. *There exists $c_0, \dots, c_9 \in \mathbb{C}$ that are not all zero, such that*

$$c_0 A_0(F(Z)) + \dots + c_7 A_7(F(Z)) + c_8 B_0(F(Z)) + c_9 B_1(F(Z)) \equiv 0, \quad (110)$$

for all $Z \in U$ when restricted on $y_7 + \sum_{i=0}^6 \mu_i y_i = 0$.

We then just need to show that (110) cannot hold by the following Lemma. This is a consequence of the following Lemma 5.34 by applying a linear change of coordinates.

Lemma 5.34. *Let $H = (h_0, \dots, h_7, g_0, \dots, g_7)$ be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (x_0, \dots, x_7, y_0, \dots, y_6) \in \mathbb{C}^{15}$ with $H(0) = 0$. Assume that H has full rank at 0. Assume that a_0, \dots, a_9 are complex numbers such that*

$$a_0 A_1(H(\tilde{z})) + \dots + a_7 A_7(H(\tilde{z})) + a_8 B_0(H(\tilde{z})) + a_9 B_1(H(\tilde{z})) = 0 \text{ for all } \tilde{z} \in U. \quad (111)$$

Then $a_i = 0$ for $1 \leq i \leq 10$.

Proof of Lemma 5.34: Suppose not. Notice that H has full rank at 0. Hence there exist 15 components \hat{H} of H that gives a local biholomorphism from \mathbb{C}^{15} to \mathbb{C}^{15} . We assume these 15 components \hat{H} are H with η being dropped, where $\eta \in \{h_0, \dots, h_7, g_0, \dots, g_7\}$. By a local biholomorphic change of coordinates, we assume $\hat{H} = (x_0, \dots, x_7, y_0, \dots, y_6)$. We still write the remaining component as η . Without loss of generality, we assume $\eta = g_7$.

First we claim the coefficient a_9 of B_1 is zero. Suppose not. Note that y_1^2, y_2^2 can be only cancelled by g_7^2 . Then g_7 will have linear y_1, y_2 terms. Hence g_7^2 will produce a $y_1 y_2$ term. It cannot be canceled by any other terms. This is a contradiction. Now we consider the coefficients of A_0, \dots, A_7 . We claim $a_i = 0, 0 \leq i \leq 7$. Suppose not. We write

$$y_7(\tilde{Z}) = \lambda_0 y_0 + \dots + \lambda_6 y_6 + \mu_0 x_0 + \dots + \mu_7 x_7 + O(2),$$

for some $\lambda_i, \mu_j \in \mathbb{C}, 0 \leq i \leq 6, 0 \leq j \leq 7$. By collecting the terms of the form $x_0 y_i$ in the Taylor expansion of (111) we get

$$a_i + a_7 \lambda_i = 0, 0 \leq i \leq 6. \quad (112)$$

By collecting the terms of the form $x_1 y_i, 0 \leq i \leq 6$, we get,

$$a_1 + a_3 \lambda_0 = 0, -a_0 + a_3 \lambda_1 = 0, -a_4 + a_3 \lambda_2 = 0, -a_7 + a_3 \lambda_3 = 0,$$

$$a_2 + a_3 \lambda_4 = 0, -a_6 + a_3 \lambda_5 = 0, a_5 + a_3 \lambda_6 = 0.$$

By collecting the terms of the form $x_2 y_i, 0 \leq i \leq 6$, we get,

$$a_2 + a_6 \lambda_0 = 0, a_4 + a_6 \lambda_1 = 0, -a_0 + a_6 \lambda_2 = 0, -a_5 + a_6 \lambda_3 = 0.$$

$$-a_1 + a_6 \lambda_4 = 0, a_3 + a_6 \lambda_5 = 0, -a_7 + a_6 \lambda_6 = 0.$$

One can further write down all the coefficients for $x_i y_j, 0 \leq i \leq 7, 0 \leq j \leq 6$. Once this is done, one easily sees that $a_i \neq 0$, for any $0 \leq i \leq 7$. Otherwise, all $a_i = 0, 0 \leq i \leq 7$.

Then by the above equations, we see that the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_1 & -a_0 & -a_4 & -a_7 & a_2 & -a_6 & a_5 \\ a_2 & a_4 & -a_0 & -a_5 & -a_1 & a_3 & -a_7 \end{pmatrix} \quad (113)$$

is of rank one. Then one can get a contradiction by, for instance, carefully checking the determinants of its 2×2 submatrices. Hence $a_i = 0, 0 \leq i \leq 7$. Finally we easily get the coefficient a_8 of B_0 is zero.

■

This establishes Proposition 5.32.

■

Remark 5.35. First fix μ_0, \dots, μ_6 with $(\sum_{i=0}^6 \mu_i^2) + 1 = 0$. By Proposition 5.32, there exists multiindices $\tilde{\beta}^1, \dots, \tilde{\beta}^{26}$, with $|\tilde{\beta}^j| \leq 11$, and

$$Z^0 = (x_0^0, \dots, x_7^0, y_0^0, \dots, y_7^0), \text{ with } \sum_{i=0}^6 \mu_i y_i + y_7 \neq 0,$$

such that

$$\begin{vmatrix} L^{\tilde{\beta}^1}(\psi_1(F)) & \dots & L^{\tilde{\beta}^1}(\psi_{26}(F)) \\ \dots & \dots & \dots \\ L^{\tilde{\beta}^{26}}(\psi_1(F)) & \dots & L^{\tilde{\beta}^{26}}(\psi_{26}(F)) \end{vmatrix} (Z^0) \neq 0.$$

We then choose $\xi^0 = (0, \dots, 0, \xi_0^0, \dots, \xi_7^0)$, where $(\xi_0^0, \dots, \xi_7^0)$ is choosen as in Lemma 5.31 associated with (y_0^0, \dots, y_7^0) . That is

$$1 + y_0^0 \xi_0^0 + \dots + y_7^0 \xi_7^0 = 0; \quad \sum_{i=0}^7 (\xi_i^0)^2 = 0, \quad \xi_j^0 = \mu_j \xi_7^0, 0 \leq j \leq 6, \quad \xi_7^0 \neq 0.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M}$.

We now define

$$\mathcal{L}_i = \frac{\partial}{\partial x_i} - \frac{\frac{\partial \rho}{\partial x_i}(z, \xi)}{\frac{\partial \rho}{\partial y_7}(Z, \xi)} \frac{\partial}{\partial y_7}, 0 \leq i \leq 7; \quad (114)$$

$$\mathcal{L}_{8+i} = \frac{\partial}{\partial y_i} - \frac{\frac{\partial \rho}{\partial y_i}(z, \xi)}{\frac{\partial \rho}{\partial y_7}(Z, \xi)} \frac{\partial}{\partial y_7}, 0 \leq i \leq 6; \quad (115)$$

for $(z, \xi) \in \mathcal{M}$ near (z^0, ξ^0) . They are tangent vector fields along \mathcal{M} . Moreover, $\frac{\partial \rho}{\partial y_n}(z, \xi)$ is nonzero near (z^0, ξ^0) .

We define for any multiindex $\alpha = (\alpha_0, \dots, \alpha_{14})$, $\mathcal{L}^\alpha = \mathcal{L}_0^{\alpha_0} \dots \mathcal{L}_{14}^{\alpha_{14}}$. Define for any 26 collection of 15-multiindices $\{\beta^1, \dots, \beta^{26}\}$,

$$\Lambda(\beta^1, \dots, \beta^{26})(z, \xi) = \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_{26}(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^{26}}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^{26}}(\psi_{26}(F)) \end{vmatrix} (z, \xi). \quad (116)$$

By the fact that $\sum_{i=0}^7 (\xi_i^0)^2 = 0$, one can check that, for any multiindex $\alpha = (\alpha_0, \dots, \alpha_{14})$, $\mathcal{L}^\alpha = L^\alpha$ when evaluated at (z^0, ξ^0) . Then as before, we get the following:

Theorem 5.36. *There exists multiindices $\{\beta^1, \dots, \beta^{26}\}$ such that*

$$\Lambda(\beta^1, \dots, \beta^{26})(z, \xi) \neq 0,$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, 0, \dots, 0)$.

6 Flattening of transversal Segre families

In this section, we verify the statement made in Hypothesis (II) for the space according to its type value. We still use the notations we have set up so far. We equip the space M with a canonical Kähler-Einstein metric ω as described before. We start with the following lemma:

Lemma 6.1. *Let $G : (M, \omega) \rightarrow (M, \omega)$ be a holomorphic isometry. In the affine space \mathcal{A} , its components consist of rational functions with bounded degrees.*

Proof of Lemma 6.1: Notice that M has been isometrically embedded into \mathbb{CP}^N through the canonical map defined before. Hence G is the restriction of a unitary transformation. Hence G can be identified with a map of the form:

$$(\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2, \dots, \tilde{\psi}_N) = \left(\sum_{j=0}^N a_{0j} \psi_j, \dots, \sum_{j=0}^N a_{ij} \psi_j, \dots, \sum_{j=0}^N a_{Nj} \psi_j \right),$$

where $\psi_0 = 1$ and (a_{ij}) is a unitary matrix. Back to the affine space \mathcal{A} , $G(z) = \left(\frac{\tilde{\psi}_1}{\tilde{\psi}_0}, \frac{\tilde{\psi}_2}{\tilde{\psi}_0}, \dots, \frac{\tilde{\psi}_N}{\tilde{\psi}_0} \right)$. Apparently $\psi_0 \neq 0$. ■

Theorem 6.2. *Suppose $\xi^0 \in \mathbb{C}^n, \xi^0 \neq (0, 0, \dots, 0)$. Then for a smooth point z^0 on the Segre variety Q_{ξ^0} and a small neighborhood $z^0 \in U \subset \mathbb{C}^n$, there is a point $z^1 \in U \cap Q_{\xi^0}$, such that Q_{z^0} and Q_{z^1} intersect transversally at ξ^0 . Moreover, there is a biholomorphic parametrization $\mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n) = (\xi_1, \xi_2, \dots, \xi_n)$, with $(\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n) \in U_1 \times U_2 \times \dots \times U_n \subset \mathbb{C}^n$, where U_j is a small neighborhood of $1 \in \mathbb{C}$, if $1 \leq j \leq 2$, and it is a small neighborhood of $0 \in \mathbb{C}$, if $3 \leq j \leq n$, with $\mathcal{G}(1, 1, 0, \dots, 0) = \xi^0$, such that $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times \dots \times U_n) \subset Q_{z^0}$, $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times \dots \times U_n) \subset Q_{z^1}$, and $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times \dots \times U_n), \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times \dots \times U_n), s \in U_1, t \in U_2$ are open pieces of Segre varieties. Also, \mathcal{G} consists of algebraic functions with total degree bounded by a constant depending only on (M, ω) .*

We first claim that, due to the invariance of the Segre family, we need only to prove the theorem for a special point $0 \neq \xi^0 \in \mathbb{C}^n \subset M$. Indeed, by the invariance property mentioned in §2, for an isometry σ , $(\sigma, \bar{\sigma})$ preserves the Segre family $\mathcal{M} \subset M \times M^*$. Here for $p \in \mathbb{P}^N$, $\bar{\sigma}(p) := \overline{\sigma(\bar{p})}$, as before.

We now proceed to the proof Theorem 6.2, by choosing a good point ξ^0 , case by case in terms of the type of the space.

Proof of Theorem 6.2: Case 1: We first assume that M is the hyperquadric. Then the defining equation for the Segre family is

$$\rho = 1 + \sum_{i=1}^n z_i \xi_i + \frac{1}{4} \left(\sum_{i=1}^n z_i^2 \right) \left(\sum_{i=1}^n \xi_i^2 \right) = 0.$$

We choose $\xi^0 = (1, 0, 0, \dots, 0)$. Hence $Q_{\xi^0} = \{z : \mathcal{F} = 1 + z_1 + \frac{1}{4}(\sum_{i=1}^n z_i^2) = 0\}$. We compute the gradient of $\mathcal{F} : \nabla \mathcal{F} = (1 + \frac{1}{2}z_1, \frac{1}{2}z_2, \dots, \frac{1}{2}z_n)$. Notice that Q_{ξ^0} is smooth at any point except

at $(-2, 0, \dots, 0)$. For $z^0 \in U \cap Q_{\xi^0}$, we can choose \tilde{z}^0, \tilde{U} such that $\tilde{U} \subset U, \tilde{z}^0 \in \tilde{U} \cap Q_{\xi^0}$, and $\tilde{U} \cap Q_{\xi^0}$ is a smooth piece of Q_{ξ^0} . Without loss of any generality we can just assume that z^0, U satisfy these properties.

Now pick $z^1 (\neq z^0) \in U \cap Q_{\xi^0}$ and compute the gradient of the defining functions of Q_{z^0} and Q_{z^1} at $\xi^0 = (a, 0, \dots, 0)$. Recall

$$Q_{z^0} = \{\xi | f_0(\xi_1, \dots, \xi_n) = 1 + \sum_{i=1}^n z_i^0 \xi_i + \frac{1}{4} (\sum_{i=1}^n (z_i^0)^2) (\sum_{i=1}^n \xi_i^2) = 0\},$$

$$Q_{z^1} = \{\xi | f_1(\xi_1, \dots, \xi_n) = 1 + \sum_{i=1}^n z_i^1 \xi_i + \frac{1}{4} (\sum_{i=1}^n (z_i^1)^2) (\sum_{i=1}^n \xi_i^2) = 0\}.$$

$$\begin{pmatrix} \nabla f_0|_{\xi^0=(1,0,\dots,0)} \\ \nabla f_1|_{\xi^0=(1,0,\dots,0)} \end{pmatrix} = \begin{pmatrix} z_1^0 + \frac{1}{2} \sum_{i=1}^n (z_i^0)^2 & z_2^0 & z_3^0 & \dots & z_n^0 \\ z_1^1 + \frac{1}{2} \sum_{i=1}^n (z_i^1)^2 & z_2^1 & z_3^1 & \dots & z_n^1 \end{pmatrix} = \begin{pmatrix} -2 - z_1^0 & z_2^0 & z_3^0 & \dots & z_n^0 \\ -2 - z_1^0 & z_2^1 & z_3^1 & \dots & z_n^1 \end{pmatrix}$$

The second equality is simplified by making use of the fact that $z^0, z^1 \in Q_{\xi^0=(1,0,\dots,0)}$, which implies that $0 = 1 + z_1^0 + \frac{1}{4} \sum_{i=1}^n (z_i^0)^2 = 1 + z_1^1 + \frac{1}{4} \sum_{i=1}^n (z_i^1)^2$. Hence,

$$\begin{aligned} \text{rank} \begin{pmatrix} \nabla f_0|_{\xi^0=(1,0,\dots,0)} \\ \nabla f_1|_{\xi^0=(1,0,\dots,0)} \end{pmatrix} &= \text{rank} \begin{pmatrix} -2 - z_1^0 & z_2^0 & \dots & z_n^0 \\ -2 - z_1^1 & z_2^1 & \dots & z_n^1 \end{pmatrix} = \text{rank} \begin{pmatrix} -2 - z_1^0 & z_2^0 & \dots & z_n^0 \\ -\Delta z_1^1 & \Delta z_2^1 & \dots & \Delta z_n^1 \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} 2 + z_1^0 & z_2^0 & \dots & z_n^0 \\ \Delta z_1^1 & \Delta z_2^1 & \dots & \Delta z_n^1 \end{pmatrix} = \text{rank} \begin{pmatrix} \Delta z_1^1 & \Delta z_2^1 & \dots & \Delta z_n^1 \\ \nabla \mathcal{F}|_{z^0} \end{pmatrix}, \end{aligned}$$

where $\Delta z_i^1 := z_i^1 - z_i^0$. Notice that z^0 is a smooth point on Q_{ξ^0} . Hence $\nabla \mathcal{F}$ is transversal to the tangent space of Q_{ξ^0} at z^0 . If we choose $z^1 \in Q_{\xi^0}$ close enough to z^0 , which ensures $(\Delta z_1^1, \dots, \Delta z_n^1)$ close enough to tangent space of Q_{ξ^0} at z^0 , we then get

$$\text{rank} \begin{pmatrix} \nabla f_0|_{\xi^0=(1,0,\dots,0)} \\ \nabla f_1|_{\xi^0=(1,0,\dots,0)} \end{pmatrix} = \text{rank} \begin{pmatrix} \Delta z_1^1 & \Delta z_2^1 & \dots & \Delta z_n^1 \\ \nabla \mathcal{F}|_{z^0} \end{pmatrix} = 2.$$

We assume that $\frac{\partial(f_0, f_1)}{\partial(\xi_1, \xi_2)} \neq 0$ at ξ^0 .

Now we introduce new variables $\tilde{\xi}_1, \dots, \tilde{\xi}_n$ and consider the following system of equations:

$$\begin{cases} P_1 : 1 + \sum_{i=1}^n (\tilde{\xi}_1 z_i^0) \xi_i + \frac{1}{4} (\sum_{i=1}^n (\tilde{\xi}_1)^2 (z_i^0)^2) (\sum_{i=1}^n \xi_i^2) &= 0 \\ P_2 : 1 + \sum_{i=1}^n (\tilde{\xi}_2 z_i^1) \xi_i + \frac{1}{4} (\sum_{i=1}^n (\tilde{\xi}_2)^2 (z_i^1)^2) (\sum_{i=1}^n \xi_i^2) &= 0 \\ P_3 : &\tilde{\xi}_3 - \xi_3 = 0 \\ \dots &\dots \\ P_n : &\tilde{\xi}_n - \xi_n = 0 \end{cases}$$

The Jacobian $\frac{\partial(P_1, \dots, P_n)}{\partial(\xi_1, \dots, \xi_n)}|_A \neq 0$ at $A = (\tilde{\xi}_1, \dots, \tilde{\xi}_n, \xi_1, \dots, \xi_n) = (1, 1, 0, \dots, 0, 1, 0, \dots, 0)$. Hence, by Theorem (A.3) in Appendix 1, we get the needed algebraic flattening. This completes the proof of Theorem (6.2) in the hyperquadric case.

Case 2. Grassmannians: Pick $\xi^0 = (\xi_{11}^0, \xi_{12}^0, \dots, \xi_{pq}^0) = (1, 0, \dots, 0)$. The defining function of the Segre family is

$$0 = 1 + z_{11}\xi_{11} + z_{12}\xi_{12} + \dots + z_{1q}\xi_{1q} + z_{21}\xi_{21} + \dots + z_{p1}\xi_{p1} + \sum_{i,j \neq 1} z_{ij}\xi_{ij} + \sum_{i,j \geq 2} (z_{11}z_{ij} - z_{i1}z_{1j})(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l) \neq (1,1)} (z_{ij}z_{kl} - z_{il}z_{jk})(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{jk}) + \text{high order terms.}$$

Then $Q_{\xi^0} = \{z | \mathcal{F} = 1 + z_{11} = 0\}$, $\nabla \mathcal{F} = (1, 0, 0, \dots, 0)$. Hence Q_{ξ^0} is smooth. For $z \in Q_{\xi^0}$, we have $z = (-1, z_{12}, \dots, z_{1q}, z_{21}, \dots, z_{p1}, \dots, z_{ij}, \dots, z_{pq})$. For $z^0, z^1 \in Q_{\xi^0}$,

$$\begin{aligned} Q_{z^0} &= \{\xi | 0 = f_0 = 1 + z_{11}^0\xi_{11} + z_{12}^0\xi_{12} + \dots + z_{1q}^0\xi_{1q} + z_{21}^0\xi_{21} + \dots + z_{p1}^0\xi_{p1} + \sum_{i,j \neq 1} z_{ij}^0\xi_{ij} + \sum_{i,j \geq 2} (z_{11}^0z_{ij}^0 - z_{i1}^0z_{1j}^0)(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l) \neq (1,1)} (z_{ij}^0z_{kl}^0 - z_{il}^0z_{jk}^0)(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{jk}) + \text{high order terms.}\} \\ Q_{z^1} &= \{\xi | 0 = f_1 = 1 + z_{11}^1\xi_{11} + z_{12}^1\xi_{12} + \dots + z_{1q}^1\xi_{1q} + z_{21}^1\xi_{21} + \dots + z_{p1}^1\xi_{p1} + \sum_{i,j \neq 1} z_{ij}^1\xi_{ij} + \sum_{i,j \geq 2} (z_{11}^1z_{ij}^1 - z_{i1}^1z_{1j}^1)(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l) \neq (1,1)} (z_{ij}^1z_{kl}^1 - z_{il}^1z_{jk}^1)(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{jk}) + \text{high order terms.}\} \end{aligned}$$

We then compute the gradient as follows:

$$\begin{aligned} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f_0}{\partial \xi_{11}} & \frac{\partial f_0}{\partial \xi_{12}} & \dots & \frac{\partial f_0}{\partial \xi_{1q}} & \frac{\partial f_0}{\partial \xi_{21}} & \dots & \frac{\partial f_0}{\partial \xi_{p1}} & \dots & \frac{\partial f_0}{\partial \xi_{pq}} \\ \frac{\partial f_1}{\partial \xi_{11}} & \frac{\partial f_1}{\partial \xi_{12}} & \dots & \frac{\partial f_1}{\partial \xi_{1q}} & \frac{\partial f_1}{\partial \xi_{21}} & \dots & \frac{\partial f_1}{\partial \xi_{p1}} & \dots & \frac{\partial f_1}{\partial \xi_{pq}} \end{pmatrix} \Big|_{\xi^0} \\ &= \begin{pmatrix} -1 & z_{12}^0 & \dots & z_{1q}^0 & z_{21}^0 & \dots & z_{p1}^0 & -z_{i1}^0 z_{1j}^0 & \dots \\ -1 & z_{12}^1 & \dots & z_{1q}^1 & z_{21}^1 & \dots & z_{p1}^1 & -z_{i1}^1 z_{1j}^1 & \dots \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\text{rank} \begin{pmatrix} \nabla f_0 \\ \nabla f_1 \end{pmatrix} = \text{rank} \begin{pmatrix} -1 & z_{12}^0 & \dots & z_{1q}^0 & z_{21}^0 & \dots & z_{p1}^0 & -z_{i1}^0 z_{1j}^0 & \dots \\ 0 & \Delta z_{12}^1 & \dots & \Delta z_{1q}^1 & \Delta z_{21}^1 & \dots & \Delta z_{p1}^1 & (-z_{i1}^0 \Delta z_{1j}^1 - z_{1j}^0 \Delta z_{i1}^1 - \Delta z_{i1}^1 \Delta z_{1j}^1) & \dots \end{pmatrix}$$

Where $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. Hence, if we choose z^1 such that $z_{12}^1 \neq z_{12}^0$, Then the rank equals to 2.

Now we introduce new variable $\tilde{\xi}_{11}, \dots, \tilde{\xi}_{pq}$ and set up the system:

$$\begin{cases} P_{11} : \det \left(I + \tilde{\xi}_{11} \cdot \begin{pmatrix} z_{11}^0 & z_{12}^0 & \dots & z_{1q}^0 \\ z_{21}^0 & z_{22}^0 & \dots & z_{2q}^0 \\ \dots & \dots & \dots & \dots \\ z_{p1}^0 & z_{p2}^0 & \dots & z_{pq}^0 \end{pmatrix} \begin{pmatrix} \xi_{11} & \xi_{21} & \dots & \xi_{p1} \\ \xi_{12} & \xi_{22} & \dots & \xi_{2p} \\ \dots & \dots & \dots & \dots \\ \xi_{q1} & \xi_{q2} & \dots & \xi_{qp} \end{pmatrix} \right) = 0 \\ P_{12} : \det \left(I + \tilde{\xi}_{12} \cdot \begin{pmatrix} z_{11}^1 & z_{12}^1 & \dots & z_{1q}^1 \\ z_{21}^1 & z_{22}^1 & \dots & z_{2q}^1 \\ \dots & \dots & \dots & \dots \\ z_{p1}^1 & z_{p2}^1 & \dots & z_{pq}^1 \end{pmatrix} \begin{pmatrix} \xi_{11} & \xi_{21} & \dots & \xi_{p1} \\ \xi_{12} & \xi_{22} & \dots & \xi_{2p} \\ \dots & \dots & \dots & \dots \\ \xi_{q1} & \xi_{q2} & \dots & \xi_{qp} \end{pmatrix} \right) = 0 \\ P_{13} : \tilde{\xi}_{13} - \xi_{13} = 0 \\ \dots \\ P_{pq} : \tilde{\xi}_{pq} - \xi_{pq} = 0 \end{cases}$$

Its Jacobian $\frac{\partial(P_{11}, \dots, P_{pq})}{\partial(\xi_{11}, \dots, \xi_{pq})}|_A \neq 0$, where $A = (\tilde{\xi}_{11}, \dots, \tilde{\xi}_{pq}, \xi_{11}, \dots, \xi_{pq}) = (1, 1, 0, \dots, 0, 1, 0, \dots, 0)$. Hence, by Theorem (A.3) in Appendix 1, we get the needed algebraic flattening.

Case 3. Symplectic Grassmannians: Pick $\xi_0 = (1, 0, 0, \dots, 0)$. The defining equation of the Segre family is

$$1 + \sum_{i=1}^n z_{ii} \xi_{ii} + 2 \sum_{i < j} z_{ij} \xi_{ij} + \sum_{2 \leq i < j} (z_{11} z_{ij} - z_{1j} z_{i1}) (\xi_{11} \xi_{ij} - \xi_{i1} \xi_{1j}) + \sum_{i < k, j < l, (i,j) \neq (1,1)} (z_{ij} z_{kl} - z_{il} z_{kj}) (\xi_{ij} \xi_{kl} - \xi_{il} \xi_{kj}) + \text{high order terms.}$$

where $z_{ji} = z_{ij}$ if $j > i$.

$Q_{\xi^0} = \{z | \mathcal{F} = 1 + z_{11} = 0\}$, $\nabla \mathcal{F} = (1, 0, \dots, 0)$. Hence Q_{ξ^0} is smooth, and for $z \in Q_{\xi^0}$ we have $z = (-1, z_{12}, z_{22}, z_{13}, \dots, z_{(n-1)n})$. For $z^0, z^1 \in Q_{\xi^0}$,

$$Q_{z^0} = \{\xi | 0 = f_0 = 1 + \sum_{i=1}^n z_{ii}^0 \xi_{ii} + 2 \sum_{i < j} z_{ij}^0 \xi_{ij} + \sum_{2 \leq i < j} (z_{11}^0 z_{ij}^0 - z_{1j}^0 z_{i1}^0) (\xi_{11} \xi_{ij} - \xi_{i1} \xi_{1j}) + \sum_{i < k, j < l, (i,j) \neq (1,1)} (z_{ij}^0 z_{kl}^0 - z_{il}^0 z_{kj}^0) (\xi_{ij} \xi_{kl} - \xi_{il} \xi_{kj}) + \text{high order terms.}\}$$

$$Q_{z^1} = \{\xi | 0 = f_1 = 1 + \sum_{i=1}^n z_{ii}^1 \xi_{ii} + 2 \sum_{i < j} z_{ij}^1 \xi_{ij} + \sum_{2 \leq i < j} (z_{11}^1 z_{ij}^1 - z_{1j}^1 z_{i1}^1) (\xi_{11} \xi_{ij} - \xi_{i1} \xi_{1j}) + \sum_{i < k, j < l, (i,j) \neq (1,1)} (z_{ij}^1 z_{kl}^1 - z_{il}^1 z_{kj}^1) (\xi_{ij} \xi_{kl} - \xi_{il} \xi_{kj}) + \text{high order terms.}\}$$

$$\begin{aligned} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f_0}{\partial \xi_{11}} & \frac{\partial f_0}{\partial \xi_{12}} & \dots & \frac{\partial f_0}{\partial \xi_{1n}} & \dots & \frac{\partial f_0}{\partial \xi_{ij}} & \dots & \frac{\partial f_0}{\partial \xi_{nn}} \\ \frac{\partial f_1}{\partial \xi_{11}} & \frac{\partial f_1}{\partial \xi_{12}} & \dots & \frac{\partial f_1}{\partial \xi_{1n}} & \dots & \frac{\partial f_1}{\partial \xi_{ij}} & \dots & \frac{\partial f_1}{\partial \xi_{nn}} \end{pmatrix} \Big|_{\xi^0} \\ &= \begin{pmatrix} -1 & z_{12}^0 & z_{13}^0 & \dots & z_{1n}^0 & \dots & -z_{1j}^0 z_{i1}^0 & \dots \\ -1 & z_{12}^1 & z_{13}^1 & \dots & z_{1n}^1 & \dots & -z_{1j}^1 z_{i1}^1 & \dots \end{pmatrix}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \text{rank} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} &= \text{rank} \begin{pmatrix} -1 & z_{12}^0 & z_{13}^0 & \dots & z_{1n}^0 & \dots & -z_{1j}^0 z_{i1}^0 & \dots \\ -1 & z_{12}^1 & z_{13}^1 & \dots & z_{1n}^1 & \dots & -z_{1j}^1 z_{i1}^1 & \dots \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} -1 & z_{12}^0 & z_{13}^0 & \dots & z_{1n}^0 & \dots & -z_{1j}^0 z_{i1}^0 & \dots \\ 0 & \Delta z_{12}^1 & \Delta z_{13}^1 & \dots & \Delta z_{1n}^1 & \dots & -z_{1j}^1 \Delta z_{i1}^1 - \Delta z_{1j}^1 z_{i1}^1 - \Delta z_{1j}^1 \Delta z_{i1}^1 & \dots \end{pmatrix} \end{aligned}$$

where $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. If we pick $z_{12}^1 \neq z_{12}^0$, then the above rank is 2. The rest of the argument is the same as in the previous case.

Case 4. Orthogonal Grassmannians: Here we use the Pfaffian embedding stated in §2. Fixing $\xi^0 = (\xi_{12}^0, \xi_{13}^0, \xi_{23}^0, \dots, \xi_{(n-1)n}^0) = (1, 0, \dots, 0)$, the defining function of the Segre family is

$$\begin{aligned} &\text{Pf} \left(\text{I} - \begin{pmatrix} 0 & z_{12}^0 & z_{13}^0 & \dots & z_{1n}^0 \\ -z_{12}^0 & 0 & z_{23}^0 & \dots & z_{2n}^0 \\ \dots & \dots & \dots & \dots & \dots \\ -z_{1n}^0 & -z_{2n}^0 & \dots & -z_{(n-1)n}^0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \xi_{12} & \xi_{13} & \dots & \xi_{1n} \\ -\xi_{12} & 0 & \xi_{23} & \dots & \xi_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ -\xi_{1n} & -\xi_{2n} & \dots & -\xi_{(n-1)n} & 0 \end{pmatrix} \right) \\ &= 1 + \sum_{i < j} z_{ij} \xi_{ij} + \sum_{2 \leq i < j} (z_{12} z_{ij} - z_{1i} z_{2j} + z_{1j} z_{2i}) (\xi_{12} \xi_{ij} - \xi_{1i} \xi_{2j} + \xi_{1j} \xi_{2i}) + \sum_{i < j < k < l, \{1,2\} \not\subset \{i,j,k,l\}} (z_{ij} z_{kl} - z_{ik} z_{jl} + z_{il} z_{jk}) (\xi_{ij} \xi_{kl} - \xi_{ik} \xi_{jl} + \xi_{il} \xi_{jk}) + \text{high order terms.} \end{aligned}$$

$Q_{\xi^0} = \{z|0 = \mathcal{F} = 1 + z_{12}\}$. Hence it is smooth. Since $z \in Q_{\xi^0}$, we have $z = (-1, z_{13}, \dots, z_{(n-1)n})$. Pick $z^0, z^1 \in Q_{\xi^0}$. Then

$$\begin{aligned} Q_{z^0} &= \{\xi|0 = f_0 = 1 + \sum_{i < j} z_{ij}^0 \xi_{ij} + \sum_{2 < i < j} (z_{12}^0 z_{ij}^0 - z_{1i}^0 z_{2j}^0 + z_{1j}^0 z_{2i}^0)(\xi_{12} \xi_{ij} - \xi_{1i} \xi_{2j} + \xi_{1j} \xi_{2i}) \\ &+ \sum_{i < j < k < l, \{1,2\} \not\subset \{i,j,k,l\}} (z_{ij}^0 z_{kl}^0 - z_{ik}^0 z_{jl}^0 + z_{il}^0 z_{jk}^0)(\xi_{ij} \xi_{kl} - \xi_{ik} \xi_{jl} + \xi_{il} \xi_{jk}) + h.o.t.s.\} \\ Q_{z^1} &= \{\xi|0 = f_1 = 1 + \sum_{i < j} z_{ij}^1 \xi_{ij} + \sum_{2 < i < j} (z_{12}^1 z_{ij}^1 - z_{1i}^1 z_{2j}^1 + z_{1j}^1 z_{2i}^1)(\xi_{12} \xi_{ij} - \xi_{1i} \xi_{2j} + \xi_{1j} \xi_{2i}) \\ &+ \sum_{i < j < k < l, \{1,2\} \not\subset \{i,j,k,l\}} (z_{ij}^1 z_{kl}^1 - z_{ik}^1 z_{jl}^1 + z_{il}^1 z_{jk}^1)(\xi_{ij} \xi_{kl} - \xi_{ik} \xi_{jl} + \xi_{il} \xi_{jk}) + h.o.t.s.\} \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} &= \begin{pmatrix} \frac{\partial f_0}{\partial \xi_{12}} & \frac{\partial f_0}{\partial \xi_{13}} & \cdots & \frac{\partial f_0}{\partial \xi_{1n}} & \cdots & \frac{\partial f_0}{\partial \xi_{ij}} & \cdots & \frac{\partial f_0}{\partial \xi_{(n-1)n}} \\ \frac{\partial f_1}{\partial \xi_{12}} & \frac{\partial f_1}{\partial \xi_{13}} & \cdots & \frac{\partial f_1}{\partial \xi_{1n}} & \cdots & \frac{\partial f_1}{\partial \xi_{ij}} & \cdots & \frac{\partial f_1}{\partial \xi_{(n-1)n}} \end{pmatrix} \Big|_{\xi^0} \\ &= \begin{pmatrix} -1 & z_{13}^0 & \cdots & z_{1n}^0 & \cdots & z_{2n}^0 & (-z_{13}^0 z_{24}^0 + z_{14}^0 z_{23}^0)a & \cdots & (-z_{1i}^0 z_{2j}^0 + z_{1j}^0 z_{2i}^0)a & \cdots \\ -1 & z_{13}^1 & \cdots & z_{1n}^1 & \cdots & z_{2n}^1 & (-z_{13}^1 z_{24}^1 + z_{14}^1 z_{23}^1)a & \cdots & (-z_{1i}^1 z_{2j}^1 + z_{1j}^1 z_{2i}^1)a & \cdots \end{pmatrix} \end{aligned}$$

$$\text{Rank} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} = \text{Rank} \begin{pmatrix} -1 & z_{13}^0 & \cdots & z_{1n}^0 & \cdots & z_{2n}^0 & \cdots \\ 0 & \Delta z_{13}^1 & \cdots & \Delta z_{1n}^1 & \cdots & \Delta z_{2n}^1 & \cdots \end{pmatrix}$$

where $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. If we choose $z_{13}^1 \neq z_{13}^0$, then the rank is 2. The rest of the argument is the same as before.

Case 5. M_{16} : Pick $\xi^0 = (\kappa_0^0, \kappa_1^0, \dots, \kappa_7^0, \eta_0^0, \eta_1^0, \dots, \eta_7^0) = (1, 0, \dots, 0)$, $z^0 \in Q_{\xi^0}$. The defining equation of the Segre family is

$$1 + x_0 \kappa_0 + x_1 \kappa_1 + \dots + x_7 \kappa_7 + y_0 \eta_0 + y_1 \eta_1 + \dots + y_7 \eta_7 + (x_0 y_0 + x_1 y_1 + \dots)(\kappa_0 \eta_0 + \kappa_1 \eta_1 + \dots) + (-y_0 x_1 + y_1 x_0 + \dots)(-\eta_0 \kappa_1 + \eta_1 \kappa_0 + \dots) + \dots + (x_0^2 + x_1^2 + \dots + x_7^2)(\kappa_0^2 + \kappa_1^2 + \dots + \kappa_7^2) + (y_0^2 + y_1^2 + \dots + y_7^2)(\eta_0^2 + \eta_1^2 + \dots + \eta_7^2) = 0.$$

$Q_{\xi^0} = \{z|\mathcal{F} = 1 + x_0 + (x_0^2 + x_1^2 + \dots + x_7^2) = 0\}$, and $\nabla \mathcal{F}|_{z^0} = (1 + 2x_0, 2x_1, \dots, 2x_7, 0, \dots, 0)$. Hence Q_{ξ^0} is smooth. For $z^0, z^1 \in Q_{\xi^0}$

$$Q_{z^0} = \{\xi|0 = f_0 = 1 + x_0^0 \kappa_0 + x_1^0 \kappa_1 + \dots + x_7^0 \kappa_7 + y_0^0 \eta_0 + y_1^0 \eta_1 + \dots + y_7^0 \eta_7 + (x_0^0 y_0^0 + x_1^0 y_1^0 + \dots)(\kappa_0 \eta_0 + \kappa_1 \eta_1 + \dots) + (-y_0^0 x_1^0 + y_1^0 x_0^0 + \dots)(-\eta_0 \kappa_1 + \eta_1 \kappa_0 + \dots) + \dots + ((x_0^0)^2 + (x_1^0)^2 + \dots + (x_7^0)^2)(\kappa_0^2 + \kappa_1^2 + \dots + \kappa_7^2) + ((y_0^0)^2 + (y_1^0)^2 + \dots + (y_7^0)^2)(\eta_0^2 + \eta_1^2 + \dots + \eta_7^2)\}$$

$$Q_{z^1} = \{\xi|0 = f_1 = 1 + x_0^1 \kappa_0 + x_1^1 \kappa_1 + \dots + x_7^1 \kappa_7 + y_0^1 \eta_0 + y_1^1 \eta_1 + \dots + y_7^1 \eta_7 + (x_0^1 y_0^1 + x_1^1 y_1^1 + \dots)(\kappa_0 \eta_0 + \kappa_1 \eta_1 + \dots) + (-y_0^1 x_1^1 + y_1^1 x_0^1 + \dots)(-\eta_0 \kappa_1 + \eta_1 \kappa_0 + \dots) + \dots + ((x_0^1)^2 + (x_1^1)^2 + \dots + (x_7^1)^2)(\kappa_0^2 + \kappa_1^2 + \dots + \kappa_7^2) + ((y_0^1)^2 + (y_1^1)^2 + \dots + (y_7^1)^2)(\eta_0^2 + \eta_1^2 + \dots + \eta_7^2)\}$$

$$\text{Rank} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} \geq \text{Rank} \begin{pmatrix} \frac{\partial f_0}{\partial \kappa_0} & \frac{\partial f_0}{\partial \kappa_1} & \cdots & \frac{\partial f_0}{\partial \kappa_7} \\ \frac{\partial f_1}{\partial \kappa_0} & \frac{\partial f_1}{\partial \kappa_1} & \cdots & \frac{\partial f_1}{\partial \kappa_7} \end{pmatrix} \Big|_{\xi^0} = \text{Rank} \begin{pmatrix} -2 - x_0^0 & x_1^0 & x_2^0 & \cdots & x_7^0 \\ -2 - x_0^1 & x_1^1 & x_2^1 & \cdots & x_7^1 \end{pmatrix}$$

Since $(-2 - x_0^0, x_1^0, x_2^0, \dots, x_7^0) \neq (0, \dots, 0)$, we can pick z^1 arbitrary close to z^0 , such that the above rank is 2. That is because Q_{ξ^0} is irreducible as we will see later and the subvarieties, defined by 2×2 minors of the last matrix in (6), are thin subsets of Q_{ξ^0} . The rest of the argument is the same as in the previous case.

Case 6. M_{27} : Take $\xi^0 = (\xi_1^0, \xi_2^0, \xi_3^0, \eta_0^0, \eta_1^0, \dots, \eta_7^0, \kappa_0^0, \kappa_1^0, \dots, \kappa_7^0, \tau_0^0, \tau_1^0, \dots, \tau_7^0) = (1, 0, \dots, 0)$. The defining function of the Segre family is $1 + r_z \cdot r_\xi$ where

$$r_z = (x_1, x_2, x_3, y_0, \dots, y_7, z_0, \dots, z_7, w_0, \dots, w_7, A, B, C, D_0, \dots, D_7, E_0, \dots, E_7, F_0, \dots, F_7, G)$$

$$r_\xi = (\xi_1, \xi_2, \xi_3, \dots, \eta_7, \dots, \kappa_7, \dots, \tau_7, A(\xi), B(\xi), C(\xi), \dots, D_7(\xi), \dots, E_7(\xi), \dots, G(\xi)).$$

Here A, B, C, D_i, E_i, F_i are homogeneous quadratic polynomials; G is a homogeneous cubic polynomial defined in Appendix 2.

For our purpose, we present terms only involving ξ_1, ξ_2 , and omit those involving $\xi_3, \eta_0, \eta_1, \dots, \eta_7, \kappa_0, \kappa_1, \dots, \kappa_7, \tau_0, \tau_1, \dots, \tau_7$ as follows: $\rho(z, \xi) = 1 + x_1\xi_1 + x_2\xi_2 + \dots + (x_1x_2 - (\sum_{i=0}^7 y_i^2))(\xi_1\xi_2 - (\sum_{i=0}^7 \tau_i^2)) + \dots$.

$Q_{\xi^0} = \{z | \mathcal{F} = 1 + x_1 = 0\}$, $\nabla \mathcal{F} = (1, 0, 0, \dots, 0)$. Hence Q_{ξ^0} is smooth and for $z \in Q_{\xi^0}$, we have $z = (-1, x_2, x_3, \dots)$. For $z^0, z^1 \in Q_{\xi^0}$,

$$Q_{z^0} = \{\xi | 0 = f_0 = 1 + x_1^0\xi_1 + x_2^0\xi_2 + \dots + (x_1^0x_2^0 - (\sum_{i=0}^7 (y_i^0)^2))(\xi_1\xi_2 - (\sum_{i=0}^7 (\tau_i^0)^2)) + \dots\}$$

$$Q_{z^1} = \{\xi | 0 = f_1 = 1 + x_1^1\xi_1 + x_2^1\xi_2 + \dots + (x_1^1x_2^1 - (\sum_{i=0}^7 (y_i^1)^2))(\xi_1\xi_2 - (\sum_{i=0}^7 (\tau_i^1)^2)) + \dots\}$$

$$\begin{aligned} \text{Rank} \begin{pmatrix} \nabla f_0|_{\xi^0} \\ \nabla f_1|_{\xi^0} \end{pmatrix} &= \text{Rank} \begin{pmatrix} \frac{\partial f_0}{\partial \xi_1} & \frac{\partial f_0}{\partial \xi_2} & \frac{\partial f_0}{\partial \xi_3} & \dots & \frac{\partial f_0}{\partial \eta_7} & \dots & \frac{\partial f_0}{\partial \kappa_7} & \dots & \frac{\partial f_0}{\partial \tau_7} \\ \frac{\partial f_1}{\partial \xi_1} & \frac{\partial f_1}{\partial \xi_2} & \frac{\partial f_1}{\partial \xi_3} & \dots & \frac{\partial f_1}{\partial \eta_7} & \dots & \frac{\partial f_1}{\partial \kappa_7} & \dots & \frac{\partial f_1}{\partial \tau_7} \end{pmatrix} \Big|_{\xi^0} \geq \text{Rank} \begin{pmatrix} \frac{\partial f_0}{\partial \xi_1} & \frac{\partial f_0}{\partial \xi_2} \\ \frac{\partial f_1}{\partial \xi_1} & \frac{\partial f_1}{\partial \xi_2} \end{pmatrix} \Big|_{\xi^0} \\ &= \text{Rank} \begin{pmatrix} -1 & -(\sum_{i=0}^7 (y_i^0)^2) \\ -1 & -(\sum_{i=0}^7 (y_i^1)^2) \end{pmatrix} \Big|_{\xi^0} \geq 2, \end{aligned}$$

for those z^1 's such that $\sum_{i=0}^7 (y_i^1)^2 \neq \sum_{i=0}^7 (y_i^0)^2$. This can be done in any small neighborhood of z^0 ; for $\{z | \sum_{i=0}^7 (y_i)^2 = B\}$ is a thin set in $\{z | 0 = 1 + x_1\}$ for each fixed $B \in \mathbb{C}$.

The rest of the argument is the same as in the previous case. This completes the proof of the flattening theorem.

■

7 Irreducibility of Segre varieties

We verify the statement in Hypothesis III in this section by proving the following theorem:

Theorem 7.1. \mathcal{M} and Q_ξ for each $\xi (\neq 0)$ are irreducible when restricted to the $\mathcal{A} \times \mathcal{A}$ and \mathcal{A} , respectively. Moreover, if $\xi \neq \tilde{\xi}$, then $Q_\xi \neq Q_{\tilde{\xi}}$.

Proof of Theorem 7.1: It suffices to show the irreducibility of Q_ξ . The latter part of the theorem is then an immediate corollary. We will prove case by case according to the type of the space.

♣1. We start with the Grassmannian space. Recall that for Z, ξ being $p \times q, q \geq p$ matrices,

$$\rho(z, \xi) = \det(I_p + Z\xi^t) = 1 + \sum_{k=1}^p \left(\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq p, 1 \leq j_1 < j_2 < \dots < j_k \leq q} Z \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \xi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix} \right). \quad (117)$$

Notice that on the right hand side of (117) there are no monomials divisible by

$$z_{ij}^2 \text{ or } z_{ik}z_{il}, z_{ik}z_{sk} \quad \forall 1 \leq i, s \leq p, 1 \leq j, k, l \leq q. \quad (118)$$

Now suppose $\rho(\cdot, \xi)$ is reducible. Write

$$\rho(z, \xi) = \det(I_p + Z\xi^t) = PQ$$

where P, Q are nonconstant polynomials in z . In the following, when we say z_{ij} appears, for instance, in P , we mean that there is a monomial in P which is divisible by z_{ij} .

Claim 1: For any fixed $1 \leq i \leq p, 1 \leq j \leq q$, z_{ij} can at most appear in one of P and Q . (That is, it cannot appear in both of them).

Proof of Claim 1: Suppose not. Write $P = \alpha_m(z_{ij})^m + \alpha_{m-1}(z_{ij})^{m-1} + \dots + \alpha_0, Q = \beta_n(z_{ij})^n + \beta_{n-1}(z_{ij})^{n-1} + \dots + \beta_0$. Here $m \geq 1, n \geq 1, \alpha$'s and β 's are polynomials in other variables with $\alpha_m \neq 0, \beta_n \neq 0$. Then z_{ij}^{m+n} appears in $\rho = PQ$. This yields a contradiction to the fact (118). Hence either $m = 0, n = 1$ or $m = 1, n = 0$. ■

Claim 2: For any $1 \leq i \leq n, 1 \leq j \neq k \leq n$, z_{ij} and z_{ik} can only appear in one of P and Q . Similarly, z_{ji} and z_{ki} can only appear in one of P and Q .

Proof of Claim 2: After Claim 1 is proved, a similar argument as used in Claim 1 yields the conclusion. ■

Write the rank of the $p \times q$ matrix ξ as k . Then ρ is of degree k in Z . We can assume k is at least two. Otherwise, ρ is linear, which is trivially irreducible. Then there is a $k \times k$ submatrix of ξ that is nonsingular. But any higher order submatrix is singular. For simplicity of notations, we assume that $\Xi \begin{pmatrix} 1 & \dots & k \\ 1 & \dots & k \end{pmatrix}$ is nonzero. Then $Z \begin{pmatrix} 1 & \dots & k \\ 1 & \dots & k \end{pmatrix}$ appears in ρ by (117). Recall by Claim 1, we know that z_{11} can only appear either in one of P and Q . Without loss of generality, assume that z_{11} appears in P only. By Claim 2, $z_{1j}, z_{j1}, 1 \leq j \leq k$, also appears in P only. Then again by Claim 2, all $z_{ij}, 1 \leq i, j \leq k$, only appear in P , namely, each variable in $Z \begin{pmatrix} 1 & \dots & k \\ 1 & \dots & k \end{pmatrix}$ only appears in P . This implies that $Z \begin{pmatrix} 1 & \dots & k \\ 1 & \dots & k \end{pmatrix}$ appears in P . Thus P has degree k in Z and Q thus has to be a constant. This is a contradiction. This establishes the irreducibility for the Type I space.

♣2. We now prove the theorem for the Type II spaces. First we notice from the definition of $\rho(z, \xi)$ and the Pfaffian the following fact:

Claim 3: For $1 \leq i < j \leq n, 1 \leq k < l \leq n, z_{ij}z_{kl}$ cannot appear in $\rho(z, \xi)$ if $\{i, j\} \cap \{k, l\} \neq \emptyset$.

Fix $\xi \in \mathcal{A} \subset G_{II}(n, n)$. Now we assume that $\rho(\cdot, \xi)$ is of degree k . We further assume $k \geq 2$. Otherwise it is trivial. For simplicity of notations, we only argue here for the case $k = 2$. The general case is of no significant difference.

By the setting above, there is a 4×4 nonsingular principal submatrix of Ξ :

$$\Xi \begin{pmatrix} i & j & k & l \\ i & j & k & l \end{pmatrix},$$

where $1 \leq i < j < k < l \leq n$. Consequently, $\rho(\cdot, \xi)$ has the quadratic term: $z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{jk}$.

Now suppose ρ is reducible. Write

$$\rho(\cdot, \xi) = PQ.$$

Here P, Q are nonconstant polynomials in z . Note P, Q both have to be of degree 1 in z . We claim that z_{ij} can only appear in either P or Q . Otherwise, $\rho(\cdot, \xi)$ will have a term that is a power of $z_{ij} : z_{ij}^m, m \geq 2$. This contradicts Claim 3. Without loss of generality, we assume z_{ij} appears in P . Then we further claim that z_{il}, z_{ik} also appear only in P . Indeed, if z_{il} or z_{ik} appears in Q , then $z_{ij}z_{il}$ or $z_{ij}z_{ik}$ appears in $\rho(\cdot, \xi) = PQ$. This again contradicts Claim 3. Furthermore, for the same reason, we conclude that z_{jk}, z_{jl} appears only in P . But then $\rho(\cdot, \xi) = PQ$ cannot have, for instance, the term $z_{il}z_{jk}$. This is a contradiction. We thus establishes that $\rho(\cdot, \xi)$ is irreducible.

♣3. We now prove the statement for the Type III spaces. Now both Z, Ξ are symmetric $n \times n$ matrices. We observe by (117) in this case there are no monomials divisible by

$$z_{ii}^2, z_{ii}z_{ij}, z_{ii}z_{ji}, 1 \leq i, j \leq n. \quad (119)$$

Then we have in the same way as in the Type I case the following:

Claim 4: $z_{ii}, 1 \leq i \leq n$, can only appear in one of P and Q , but not both.

Claim 5: If z_{ii} appears in P (or Q), then all $z_{il}, 1 \leq l \leq n$, can only appear in P (or Q).

We let again the degree of $\rho(\cdot, \xi)$ in z be k . We only argue for the case with $k = 2$ and the other can be similarly done. Then $\rho(\cdot, \xi)$ has a nonzero quadratic term. We divide our arguments into two cases:

Case I: Assume that we can find a quadratic term $z_{ij}z_{kl}$ ($i \leq j, k \leq l$) in $\rho(\cdot, \xi)$ such that either $i = j$ or $k = l$. In this case, we assume, for simplicity, that $k = l = n$. That is, $\rho(\cdot, \xi)$ has the term $z_{ij}z_{nn}$. Then by Lemma 5.9, we conclude that $z_{in}z_{jn}$ also appears in $\rho(\cdot, \xi)$. Suppose ρ is reducible. Again write

$$\rho = PQ.$$

Here P, Q are nonconstant polynomials in z . Notice that P, Q both have to be of degree 1 in z . We write

$$P = 1 + \lambda_1 z_{nn} + \dots,$$

$$Q = 1 + \lambda_2 z_{ij} + \dots,$$

with $\lambda_i \neq 0, 1 \leq i \leq 2$. By Claim 5, we conclude that z_{in}, z_{jn} cannot appear in Q . This is a contradiction since then PQ cannot produce $z_{in}z_{jn}$.

Case II: We consider the case where we cannot find $z_{ij}z_{kl}$ in $\rho(\cdot, \xi)$ such that $i = j$ or $k = l$. For simplicity, we assume $k = n - 1, l = n$, i.e., ρ has the term $z_{ij}z_{(n-1)n}, 1 \leq i < j \leq n$. Then by Lemma 5.12, ρ has either $z_{i(n-1)}z_{jn}$ or $z_{j(n-1)}z_{in}$. Without loss of generality, we assume ρ has the term $z_{i(n-1)}z_{jn}$. Now suppose ρ is reducible. Again write

$$\rho = PQ.$$

Here P, Q are nonconstant polynomials in z . Note P, Q both have to be of degree 1 in z . Write

$$P = 1 + \lambda_1 z_{(n-1)n} + \dots,$$

$$Q = 1 + \lambda_2 z_{ij} + \dots,$$

with $\lambda_i \neq 0, 1 \leq i \leq 2$. We claim that Q cannot have $z_{(n-1)n}$. Otherwise, $\rho = PQ$ will have the quadratic term $z_{(n-1)n}^2$. Then by Lemma 5.9, ρ also has the term $z_{(n-1)(n-1)}z_{nn}$. This contradicts the hypothesis of Case II. Now since ρ has the term $z_{i(n-1)}z_{jn}$, then Q has either $z_{i(n-1)}$ or z_{jn} term. Without loss of generality, we assume Q has the $z_{i(n-1)}$ term. Then $\rho = PQ$ will have a $z_{i(n-1)}z_{(n-1)n}$ term. Then Lemma 5.10 yields that ρ also has a $z_{in}z_{(n-1)(n-1)}$ term. This again contradicts the hypothesis of Case II. Hence Case II cannot happen in our setting.

This proves the case $k = 2$. By applying (a slightly generalized version of) Lemma 5.9-5.12, The general case $k \geq 3$ can be established without any essential difference.

♣4. We now prove the theorem for the Type IV spaces. In this case,

$$\rho(Z, \xi) = 1 + z_1 \xi_1 + \dots + z_n \xi_n + \frac{1}{4} \left(\sum_{i=1}^n z_i^2 \right) \left(\sum_{i=1}^n \xi_i^2 \right).$$

Recall for the irreducible hyperquaric $Q^n, n \geq 3$. Suppose ρ is reducible. Again write

$$\rho = PQ.$$

Here P, Q are nonconstant polynomials in z . Note P, Q both have to be of degree 1 in z . Now write $A = \frac{1}{4}(\sum_{i=1}^n \xi_i^2)$. We can assume $A \neq 0$. Otherwise, it is trivially irreducible. We can then write

$$P = 1 + \mu_1 z_1 + \mu_2 z_2 + \dots + \mu_n z_n;$$

$$Q = 1 + \frac{A}{\mu_1} z_1 + \frac{A}{\mu_2} z_2 + \dots + \frac{A}{\mu_n} z_n,$$

where $\mu_i \neq 0, 1 \leq i \leq n$. This is due to the fact that the coefficient of z_i^2 in ρ is A . Now since for each $1 \leq i < j \leq n$, the coefficient of $z_i z_j$ in ρ is 0. We have,

$$\mu_i^2 = -\mu_j^2 \text{ for all } 1 \leq i < j \leq n.$$

This is a contradiction since $n \geq 3$. This establishes the desired irreducibility for the Type IV space.

♣5. We then prove the statement for the 16-dimensional exceptional space case. In this case, for fixed $\xi = (\eta_0, \dots, \eta_7, \zeta_0, \dots, \zeta_7) \neq 0$, $\rho(z, \xi)$ is a quadratic polynomial in $z := (x, y) := (x_0, \dots, x_7, y_0, \dots, y_7) \in \mathbb{C}^{16}$.

$$\rho(z, \xi) = 1 + \eta_0 x_0 + \dots + \eta_7 x_7 + \zeta_0 y_0 + \dots + \zeta_7 y_7 + a_0 A_0(x, y) + \dots + a_7 A_7(x, y) + b_0 B_0(x) + b_1 B_1(y), \quad (120)$$

for some $a_0, \dots, a_7, b_0, b_1 \in \mathbb{C}$. Here for the expressions of $A_0, \dots, A_7, B_0, B_1$, see Appendix 2.

If all $a_0, \dots, a_7, b_0, b_1$ are zero, then $\rho(\cdot, \xi)$ is trivially irreducible. We now assume one of them is nonzero. Thus $\rho(\cdot, \xi)$ is quadratic. Now we first prove if either b_0 or b_1 is nonzero, then $\rho(\cdot, \xi)$ is irreducible. Suppose not. Write

$$\rho(\cdot, \xi) = PQ.$$

Here P, Q are nonconstant polynomials in z . Note P, Q are both of degree 1. Without loss of generality, we assume that $b_0 \neq 0$. Then by the expression of ρ , we are able to write

$$P = 1 + \mu_0 x_0 + \dots + \mu_7 x_7 + L_1(y)$$

$$Q = 1 + \frac{b_0}{\mu_0} x_0 + \dots + \frac{b_0}{\mu_7} x_7 + L_2(y),$$

where $L_1(y), L_2(y)$ are linear homogeneous in y . Here all $\mu_i \neq 0, 1 \leq i \leq 7$. Note the fact that there are no terms of the form $x_i x_j, 1 \leq i < j \leq 7$, in $\rho(\cdot, \xi)$. This implies that

$$\mu_i^2 = -\mu_j^2, \text{ for all } 0 \leq i < j \leq 7.$$

This is trivially impossible.

We thus only need to consider the case where $b_0 = b_1 = 0$. Then a_0, \dots, a_7 are not all zero. Without loss of generality, we assume $a_0 \neq 0$. Suppose ρ is reducible. Write

$$\rho(\cdot, \xi) = PQ.$$

Here P, Q are nonconstant polynomials in z . Note P, Q both have to be of degree 1 in z . Note that $\rho(\cdot, \xi)$ does not have $x_i x_j, y_i y_j, 0 \leq i, j \leq 7$ terms. We can thus assume

$$P = 1 + \mu_0 x_0 + \dots + \mu_7 x_7,$$

$$Q = 1 + \frac{a_0}{\mu_0} y_0 + \dots + \frac{a_0}{\mu_7} y_7,$$

with all $\mu_i \neq 0, 0 \leq i \leq 7$. Note that for $0 \leq i < j \leq 7$, the terms $x_i y_j, x_j y_i$ have the same coefficients with different signs in $\rho(\cdot, \xi)$, by the expression of $A_l, 1 \leq l \leq 7$. This again implies

$$\mu_i^2 = -\mu_j^2, \text{ for all } 0 \leq i < j \leq 7.$$

It is thus a contradiction.

♣6. We finally prove the theorem for the 27-dimensional exceptional space case. In this case, for fixed $\xi = (\alpha_1, \alpha_2, \alpha_3, \gamma_0, \dots, \gamma_7, \eta_0, \dots, \eta_7, \zeta_0, \dots, \zeta_7) \neq 0$, $\rho(z, \xi)$ is a cubic polynomial in $z := (x, y, t, w) := (x_1, x_2, x_3, y_0, \dots, y_7, t_0, \dots, t_7, w_0, \dots, w_7) \in \mathbb{C}^{27}$.

$$\begin{aligned} \rho(z, \xi) = & 1 + \alpha_1 x_1 + \dots + \alpha_3 x_3 + \sum_{i=0}^7 \gamma_i y_i + \sum_{i=0}^7 \eta_i t_i + \sum_{i=0}^7 \zeta_i w_i \\ & + aA(z) + bB(z) + cC(z) + \sum_{i=0}^7 d_i D_i(z) + \sum_{i=0}^7 e_i E_i(z) + \sum_{i=0}^7 f_i F_i(z) + gG(z), \end{aligned}$$

for some $a, b, c, d_i, e_i, f_i, g \in \mathbb{C}, 0 \leq i \leq 7$. Here for the expressions of $A, B, C, D_i, E_i, F_i, G$, see Appendix 2.

If all $a, b, c, d_i, e_i, f_i, g, 0 \leq i \leq 7$, are zero, then $\rho(\cdot, \xi)$ is trivially irreducible. Now assume one of them is nonzero. We first consider the case if a, b, c are not all zero. Without loss of generality, we assume that $c \neq 0$. Now suppose ρ is reducible. Write

$$\rho(\cdot, \xi) = PQ.$$

Here P, Q are nonconstant polynomials in z . Since ρ has $x_1 x_2$ term, then x_1, x_2 will appear in P or Q . But note there is no monomial divisible by x_i^2 in ρ for each $1 \leq i \leq 3$. Thus x_i can only appear in one of P, Q . If the product $x_1 x_2$ appears in one of P, Q , without loss of generality, we assume it appears in P . Write $P = 1 + \lambda_1 x_1 x_2 + \dots$, with $\lambda_1 \neq 0$. Then Q is of degree 1. This implies ρ is of degree 3 in z . Consequently, g is nonzero. Then ρ has all terms in G , in particular, it has the term $x_1 w^2$. We claim that Q can have only the x_3 term besides the constant 1, i.e.,

$$Q = 1 + \lambda_2 x_3,$$

with $\lambda_2 \neq 0$. This is due to the fact that the only monomials divisible by $x_1 x_2$ in ρ is $x_1 x_2, x_1 x_2 x_3$. This is then a contradiction since PQ cannot produce $x_1 w^2$.

Now we study the case when x_1, x_2 appear separately in P, Q . Assume that

$$P = 1 + \lambda_1 x_1 + \dots,$$

$$Q = 1 + \lambda_2 x_2 + \dots,$$

where $\lambda_i \neq 0$. Since there are no monomials divisible by the $x_1 y_i, x_2 y_i, 0 \leq i \leq 7$, terms in ρ , thus P, Q cannot have any term involving $y_i, 0 \leq i \leq 7$. This is a contradiction since $cC(z)$ produces a y_0^2 term, but it cannot be produced by PQ .

We now then consider the case where $a = b = c = 0$. Then d_i, e_i, f_i, g are not all zero. We first consider the case that $d_i, e_i, f_i, 0 \leq i \leq 7$, are not all zero. Without loss of generality, we assume $d_0 \neq 0$. If the product $x_3 y_0$ appears in one of P, Q , assume

$$P = 1 + \lambda_1 x_3 y_0 + \dots,$$

with $\lambda_1 \neq 0$. Then Q is of degree 1. Note the only monomials divisible by x_3y_0 in ρ is $x_3y_0, x_3y_0^2$. Then Q can only be of form:

$$Q = 1 + \lambda_2 y_0,$$

with $\lambda_2 \neq 0$. This implies $g \neq 0$. Thus $\rho(z, \xi)$ also has the $x_1x_2x_3$ term. But this is again a contradiction since it cannot be produced by PQ . Now if x_3 and y_0 appear separately in P and Q , then we assume,

$$P = 1 + \lambda_1 x_3 + \dots,$$

$$Q = 1 + \lambda_2 y_0 + \dots,$$

with $\lambda_i \neq 0, 1 \leq i \leq 2$. Now since ρ has no terms divisible by $x_3t_i, x_3w_i, 0 \leq i \leq 7$, Q has no terms involving t_0 and w_0 , and

$$P = 1 + \lambda_1 x_3 + \lambda_3 t_0 w_0 + \dots,$$

with $\lambda_3 \neq 0$. Then $\rho = PQ$ will have the term $t_0 w_0 y_0$. This implies $g \neq 0$. Then ρ has the term $x_1x_2x_3$. But note that the only terms divisible by $t_0 w_0$ in ρ are $t_0 w_0$ and $t_0 w_0 y_0$. This implies

$$Q = 1 + \lambda_2 y_0.$$

This is again a contradiction since PQ cannot produce $x_1x_2x_3$. Finally we consider the case that $a = b = c = 0, d_i = e_i = f_i = 0, 0 \leq i \leq 7, g \neq 0$. This can be done similarly as before by analyzing the distribution of terms in P and Q , whose details we skip here.

■

As an application of the irreducibility of ρ , we have the following, which was used before:

Proposition 7.2. *Suppose U is an connected open set in \mathbb{C}^n . Then the Segre family \mathcal{M} restricted to $U \times \mathbb{C}^n$ is an irreducible analytic variety.*

Proof of Proposition 7.2: Recall that \mathcal{M} is irreducible in $\mathbb{C}^n \times \mathbb{C}^n$, $\mathcal{M}_{\text{sing}} = \{(z, \xi) : \frac{\partial \rho}{\partial \xi_j} = 0, \frac{\partial \rho}{\partial z_j} = 0, \forall j\}$, $\mathcal{M}_{\text{reg}} = \mathcal{M} \setminus \mathcal{M}_{\text{sing}}$. Define $\mathcal{M}_{\text{SING}} = \{(z, \xi) : \frac{\partial \rho}{\partial \xi_j} = 0, \forall j\} \cup \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}$, and $\mathcal{M}_{\text{REG}} = \mathcal{M} \setminus \mathcal{M}_{\text{SING}}$. Notice that $\mathcal{M}_{\text{sing}} \subset \mathcal{M}_{\text{SING}}$ and \mathcal{M}_{REG} is a Zariski open subset of \mathcal{M}_{reg} .

Since $\rho(z, \xi)$ is an irreducible polynomial and $\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial z_j}, j = 1, \dots, n$ are polynomials with lower degrees, $\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial z_j}, j = 1, \dots, n$ are not identically zero on $\mathcal{M} = \{\rho(z, \xi) = 0\}$. Each of $\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial z_j}$ defines a proper subvariety inside \mathcal{M} .

From the proof of Theorem 6.2, we know for each $\tilde{z} \in \mathbb{C}^n$, there is a certain point $\tilde{\xi}$ on $Q_{\tilde{z}}$ such that at least one derivative in ξ at $(\tilde{z}, \tilde{\xi})$ does not vanish. Hence $\mathcal{M}_{\text{SING}}$ does not contain any Segre variety. Also the projection of \mathcal{M}_{REG} to $\tilde{z} = (z_1, \dots, z_n)$ at $(\tilde{z}, \tilde{\xi}) \in \mathcal{M}_{\text{REG}}$ is a submersion. Since Q_z is irreducible for $z \in \mathbb{C}^n \setminus (0, \dots, 0)$, $Q_z \cap \mathcal{M}_{\text{REG}}$ is connected.

To prove the theorem, we just need to show that $\mathcal{M}_{\text{REG}}|_{U \times \mathbb{C}^n} = \mathcal{M}_{\text{reg}}|_{U \times \mathbb{C}^n} \setminus (\{\frac{\partial \rho}{\partial \xi_j} = 0, \forall j\} \cup \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\})$ is connected. Write the above projection map to z as $\Phi : \mathcal{M}_{\text{REG}}|_{U \times \mathbb{C}^n} \rightarrow \mathbb{C}^n$. Since it is a submersion, it is an open mapping. Suppose z^0 is a point in U , which is not $(0, \dots, 0)$. As mentioned above, we know that each fiber of Φ is connected. For any $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}}$ in the fiber above z^0 , we can choose a connected neighborhood V of (z^0, ξ^0) on $\mathcal{M}_{\text{REG}}|_{U \times \mathbb{C}^n}$ such that $\Phi(V)$ is neighborhood of z_0 . Hence, for any $z \in \Phi(V)$, any point in $Q_z \cap \mathcal{M}_{\text{REG}}$ can be connected by a smooth curve inside $\mathcal{M}_{\text{REG}}|_{V \times \mathbb{C}^n}$ to (z_0, ξ_0) . Since U is connected, by a standard open-closeness argument, we see that $\mathcal{M}_{\text{REG}}|_{U \times \mathbb{C}^n}$ is connected.

■

A Appendix 1: A generalization of the classical Hurwitz theorem

In this appendix, we present the following generalization of the classical Hurwitz theorem (see the book of Bochner-Martin [BM] for the classical Hurwitz theorem), which played an important role in the proof of our main theorem.

Theorem A.1. *Suppose G is a holomorphic function defined over $U \times V \times W \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^m$, where U, V are connected open subset of \mathbb{C} and W is a connected open subset of \mathbb{C}^m . Use $(s, t, \xi_1, \dots, \xi_m) = (s, t, \xi)$ for the coordinates of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^m$. Suppose that G is a rational function of t, ξ_1, \dots, ξ_m of degree less than N for each fixed $s \in U$. And suppose that G is a rational function of s, ξ_1, \dots, ξ_m of degree less than N for each fixed $t \in V$. Then G is a rational function on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^m$ of (s, t, ξ) of degree less than N .*

Proof of Theorem A1: We basically follow the original argument presented in [BM] with certain needed modifications.

If G is identically zero, there is nothing to prove. We assume that G is not identically zero. Then we can choose $(s_0, t_0, \xi_0) \in U \times V \times W$ such that $G(s_0, t_0, \xi_0) \neq 0$. Shrinking the domain if necessary, we can assume, without loss of generality, that $G \neq 0$ in $U \times V \times W$.

Since G is a rational function of t, ξ_1, \dots, ξ_m for an arbitrary $s \in U$ with degree less than N , we can write G as $G(s, t, \xi) = \frac{\sum_{|I|+|J|<N} C_{IJ}(s) \xi^I t^J}{\sum_{|I|+|J|<N} B_{IJ}(s) \xi^I t^J}$ for any $s \in U$. Here, as usual, $\xi^I := \xi_1^{i_1} \xi_2^{i_2} \dots \xi_m^{i_m}$ for $I = (i_1, \dots, i_m)$. By the assumption, $G(s, t, \xi)$ is also a rational function of (t, ξ) for each fixed s , and thus we can write

$$G(s, t, \xi) = \frac{\sum_{|J|<N} \{ \sum_{|I|<N-|J|} C_{IJ}(s) \xi^I \} t^J}{\sum_{|J|<N} \{ \sum_{|I|<N-|J|} B_{IJ}(s) \xi^I \} t^J} \quad (\text{A1})$$

for any $s \in U$

Because $G \neq 0$ we have $\sum_J \{ | \sum_{|I|<N-|J|} B_{IJ}(s) \xi^I | + | \sum_{|I|<N-|J|} C_{IJ}(s) \xi^I | \} > 0$ for $s \in U$ and $\xi \in W$. From (A1), we have

$$\sum_{|J|<N} \{ \sum_{|I|<N-|J|} B_{IJ}(s) \xi^I \} (t^J G(s, t, \xi)) - \sum_{|J|<N} \{ \sum_{|I|<N-|J|} C_{IJ}(s) \xi^I \} t^J = 0 \quad (\text{A2})$$

For convenience, we denote $\{t^J G(s, t, \xi)\}_J, \{t^J\}_J$ by $\{\phi_1, \dots, \phi_M\}$ and denote $\{\sum_{|I|<N-|J|} B_{IJ}(s) \xi^I\}_J, \{-\sum_{|I|<N-|J|} C_{IJ}(s) \xi^I\}_J$ by $\{\psi_1, \dots, \psi_M\}$.

Now substituting t by $t_1, \dots, t_{M-1}, t \in V$, we get

$$\begin{pmatrix} \phi_1(s, t_1, \xi) & \phi_2(s, t_1, \xi) & \phi_3(s, t_1, \xi) & \dots & \phi_M(s, t_1, \xi) \\ \dots & \dots & \dots & \dots & \dots \\ \phi_1(s, t_{M-1}, \xi) & \phi_2(s, t_{M-1}, \xi) & \phi_3(s, t_{M-1}, \xi) & \dots & \phi_M(s, t_{M-1}, \xi) \\ \phi_1(s, t, \xi) & \phi_2(s, t, \xi) & \phi_3(s, t, \xi) & \dots & \phi_M(s, t, \xi) \end{pmatrix} \begin{pmatrix} \psi_1(s, \xi) \\ \psi_2(s, \xi) \\ \dots \\ \psi_M(s, \xi) \end{pmatrix} = 0 \quad (\text{121})$$

Since $(\psi_1(s, \xi), \dots, \psi_M(s, \xi))$ is a non zero vector for any $s \in U, \xi \in W$ we have

$$\det \begin{pmatrix} \phi_1(s, t_1, \xi) & \dots & \phi_M(s, t_1, \xi) \\ \vdots & \ddots & \vdots \\ \phi_1(s, t_{M-1}, \xi) & \dots & \phi_M(s, t_{M-1}, \xi) \\ \phi_1(s, t, \xi) & \dots & \phi_M(s, t, \xi) \end{pmatrix} \equiv 0$$

Applying the Laplacian expansion through the last row, we get

$$\sum_{i=1}^M \Phi_i(t_1, \dots, t_{M-1}, \xi, s) \phi_i(s, t, \xi) \equiv 0$$

where $\{\Phi_i\}$ are the corresponding $(M-1) \times (M-1)$ minors of the above $M \times M$ matrix.

Suppose we can choose some $t_1, t_2, \dots, t_{M-1} \in V$ such that at least one of $\Phi'_j s$, as functions of (ξ, s) , is not identically zero. We then claim that for one of the non-zero $\Phi'_i s$, its associated ϕ_i takes the form: $\phi_i(s, t, \xi) = t^J G(s, t, \xi)$ for some J .

Indeed, if this is not the case, we have a relation as follows:

$$0 \equiv \sum_{j=1}^{M'} \Phi_j(t_1, \dots, t_{M-1}, \xi, s) \phi_j(s, t, \xi) = \sum_{j=1}^{M'} \Phi_j(t_1, \dots, t_{M-1}, \xi, s) t^{J_j}$$

where at least one Φ_j is not identically zero and J_j is different for different j . As a polynomial of t , we conclude that each $\Phi_j \equiv 0$, that is a contradiction to our assumption.

Therefore, we see that

$$\left(\sum_i \Phi_i(t_1, \dots, t_{M-1}, \xi, s) t^{J_i} \right) G(s, t, \xi) + \sum \Phi_j(t_1, \dots, t_{M-1}, \xi, s) t^{J_j} \equiv 0$$

and the coefficient of G is not identically zero. By the hypothesis, G is a rational function of (s, ξ) for each fixed t . Noticing that each Φ_i is a polynomial function in $\{t^J G(s, t, \xi)\}_J, \{t^J\}_J$ with t taking values in $\{t_1, \dots, t_{M-1}\}$, we conclude that each Φ_i is a rational function of (s, ξ) . Hence, G is a rational function of (s, t, ξ)

If each $\Phi_i(t_1, \dots, t_{M-1}, s, \xi)$ is identically zero for any $t_1, \dots, t_{M-1} \in V$, we see that each $(M-1) \times (M-1)$ minor of the determinant is identically zero for any $t_1, \dots, t_{M-1}, s, \xi$. On the other hand, we know that the first row vector in the matrix is a not a zero vector, we can find a certain $d \times d$ -submatrix of the coefficient matrix (121) such that not all its minors when expanding along the last row are identically zero for some values $(t_1, \dots, t_{d-1}) \in V$. Then by an induction argument, we can reduce the proof to the case with $d = (M-1) \times (M-1)$.

■

Definition A.2. Suppose F is an algebraic function defined on $\xi \in \mathbb{C}^n$. The total degree of F is defined to be the total degree of its minimum polynomial. Namely, let $P(\xi; X)$ be an irreducible minimum polynomial of F , the total degree of F is defined as the degree of $P(\xi; X)$ as a polynomial in (ξ, X) .

In what follows, for an algebraic function ϕ , we write $\deg \phi$ for its total degree. We next present a lemma about algebraic functions, whose proof is more or less standard (See, for instance, [Fang]):

Lemma A.3. 1. Suppose ϕ_1, ϕ_2 are algebraic functions defined in $\xi \in U \subset \mathbb{C}^n$ with $\deg \phi_1, \deg \phi_2 < N$. Then $\phi_1 \pm \phi_2, \phi_1 \phi_2, 1/\phi_1$ (if $\phi_1 \not\equiv 0$) are algebraic functions and their degrees are bounded above by a constant depending only on N, n .

2. Suppose $\phi_1(z_1, \dots, z_n)$ is an algebraic function of total degree bounded by N , and suppose that $\psi_1(\xi_1, \dots, \xi_m), \dots, \psi_n(\xi_1, \dots, \xi_m)$ are algebraic functions with total degree bounded by N as well. Let

$$A_0 = (\xi_1^0, \xi_2^0, \dots, \xi_m^0) \in \mathbb{C}^m,$$

where ψ_1, \dots, ψ_n are holomorphic functions in a neighborhood of A_0 and let ϕ_1 be a holomorphic in a neighborhood $U \subset \mathbb{C}^n$ of $(\psi_1(A_0), \psi_2(A_0), \dots, \psi_m(A_0))$. Then the composition $\Phi(\xi_1, \dots, \xi_m) = \phi_1(\psi_1(\xi_1, \dots, \xi_m), \psi_2(\xi_1, \dots, \xi_m), \psi_3(\xi_1, \dots, \xi_m), \dots, \psi_n(\xi_1, \dots, \xi_m))$ is an algebraic function with total degree bounded by a constant $C(N, n, m)$ depending only on (N, n, m) .

3. Suppose $P_1(z_1, z_2, \dots, z_m, \xi_1, \xi_2, \dots, \xi_n), \dots, P_n(z_1, z_2, \dots, z_m, \xi_1, \xi_2, \dots, \xi_n)$ are algebraic functions with total degrees bounded from above by Nm which are holomorphic in a neighborhood $U \times V \subset \mathbb{C}^m \times \mathbb{C}^n$ of $A_0 = (z_1^0, \dots, z_m^0, \xi_1^0, \dots, \xi_n^0)$. Suppose that

$$\begin{cases} P_1(z_1, z_2, \dots, z_m, \xi_1, \dots, \xi_n) = 0 \\ P_2(z_1, z_2, \dots, z_m, \xi_1, \dots, \xi_n) = 0 \\ \dots \\ P_n(z_1, z_2, \dots, z_m, \xi_1, \dots, \xi_n) = 0 \end{cases}$$

has a solution $A_0 = (z_1^0, \dots, z_m^0, \xi_1^0, \dots, \xi_n^0)$, and the Jacobian $\frac{\partial(P_1, P_2, \dots, P_n)}{\partial(\xi_1, \xi_2, \dots, \xi_n)}(z_1^0, z_2^0, \dots, z_m^0, \xi_1^0, \dots, \xi_n^0) \neq 0$. Then $\xi_1 = \phi_1(z_1, z_2, \dots, z_m), \xi_2 = \phi_2(z_1, z_2, \dots, z_m), \dots, \xi_n = \phi_n(z_1, z_2, \dots, z_m)$ in a neighborhood of $\tilde{U} \subset U \subset \mathbb{C}^m$ and ϕ_1, \dots, ϕ_n are algebraic functions with total degrees bounded by $C(N, n, m)$.

Now we give an algebraic version of the Hurwitz theorem with degree controlled.

Theorem A.4. Let $F(s, t, \xi_1, \xi_2, \dots, \xi_m)$ be holomorphic over $U \times V \times W \subset \mathbb{C}^{m+2}$. Suppose that for any fixed $s \in U \subset \mathbb{C}$, F is an algebraic function in (t, ξ_1, \dots, ξ_m) with its total degree uniformly bounded by N ; and for any fixed $t \in V \subset \mathbb{C}$, F is an algebraic function of (s, ξ_1, \dots, ξ_m) with its total degree uniformly bounded by N . Then F is an algebraic function with total degree bounded by a constant depending only on (m, N) .

Proof of Theorem A.4: We assume, without loss of generality, that $F \neq 0$ over $U \times V \times W$. For each fixed $t \in V$ we have $C_{IJK}(t)$, such that

$$\sum_{|I|+|J|+|K| \leq N} C_{IJK}(t) s^I \xi^J (F(s, t, \xi))^K = 0$$

Since $F \neq 0$, we can make $\sum_{IK} |\sum_J C_{IJK}(t) \xi^J| > 0$

We denote $\{s^I F(s, t, \xi)^K\}_{I,K}$ by $\{\psi_i(s, t, \xi)\}_{i=1}^M$ and $\{\sum_J C_{IJK}(t) \xi^J\}$ by $\phi_1(t, \xi), \dots, \phi_M(t, \xi)$ as before. For any $s_1, \dots, s_{M-1} \in U$, we have the following:

$$\begin{pmatrix} \psi_1(s_1, t, \xi) & \psi_2(s_1, t, \xi) & \dots & \psi_M(s_1, t, \xi) \\ \psi_1(s_2, t, \xi) & \psi_2(s_2, t, \xi) & \dots & \psi_M(s_2, t, \xi) \\ \dots & \dots & \dots & \dots \\ \psi_1(s_{M-1}, t, \xi) & \psi_2(s_{M-1}, t, \xi) & \dots & \psi_M(s_{M-1}, t, \xi) \\ \psi_1(s, t, \xi) & \psi_2(s, t, \xi) & \dots & \psi_M(s, t, \xi) \end{pmatrix} \begin{pmatrix} \phi_1(t, \xi) \\ \phi_2(t, \xi) \\ \dots \\ \phi_{M-1}(t, \xi) \\ \phi_M(t, \xi) \end{pmatrix} = 0$$

As argued before, we have

$$\begin{vmatrix} \psi_1(s_1, t, \xi) & \psi_2(s_1, t, \xi) & \dots & \psi_M(s_1, t, \xi) \\ \psi_1(s_2, t, \xi) & \psi_2(s_2, t, \xi) & \dots & \psi_M(s_2, t, \xi) \\ \dots & \dots & \dots & \dots \\ \psi_1(s_{M-1}, t, \xi) & \psi_2(s_{M-1}, t, \xi) & \dots & \psi_M(s_{M-1}, t, \xi) \\ \psi_1(s, t, \xi) & \psi_2(s, t, \xi) & \dots & \psi_M(s, t, \xi) \end{vmatrix} \equiv 0 \quad (122)$$

where s_1, \dots, s_{M-1} are arbitrary numbers in U . Applying the Laplacian expansion along the last row, we have the following:

$$\sum_{i=1}^M \Psi_i(s_1, s_2, \dots, s_{M-1}, t, \xi) \psi_i(s, t, \xi) \equiv 0$$

Here $\Psi_i(s_1, s_2, \dots, s_{M-1}, t, \xi)$ is the $(M-1) \times (M-1)$ minor in (122) obtained by deleting its i -th column and M -th row. Since for fixed s_1, \dots, s_{M-1} , $\psi_i(s_j, t, \xi)$ $i = 1, \dots, M$ $j = 1, \dots, M-1$ are algebraic functions of t, ξ with degree bounded by N and M is determined by N . By Lemma A.3 we conclude that Ψ_i $i = 1, \dots, M$ are algebraic functions of (t, ξ) with total degree bounded by a constant $C(N, m)$, depending only on (N, m) .

Now suppose we can find some s_1, s_2, \dots, s_{M-1} such that at least one of Ψ_i is not identically zero as a function of t, ξ . Then by a similar argument as in the proof of the classical Hurwitz Theorem, we can assume without loss of generality that at least one of $\Psi_i(s_1, s_2, \dots, s_{M-1}, t, \xi)$, which corresponds to one of $\{s^I F(s, t, \xi)^K\}_{I,K, |K| \geq 1}$, is not identically zero. Hence, we have

$$\sum_{i=1}^M \Psi_i(s_1, s_2, \dots, s_{M-1}, t, \xi) \psi_i(s, t, \xi) = \sum_{|I|+|K| \leq N} f_{IK}(s_1, s_2, \dots, s_{M-1}, t, \xi) s^I F(s, t, \xi)^K = 0$$

Here s_1, \dots, s_{M-1} are fixed numbers chosen from U and $f_{IK}(s_1, s_2, \dots, s_{M-1}, t, \xi)$ are just a different notation for the same quantity $\Psi_i(s_1, s_2, \dots, s_{M-1}, t, \xi)$. They are algebraic functions with degree bounded from above by $C(N, m)$. And at least one of f_{IK} with $|K| \geq 1$ is not identically zero as a function of t, ξ . Since for each I, K with $|I| + |K| \leq N$, f_{IK} is algebraic function with total degree bounded by $C(N, m)$, we can find an irreducible polynomial, whose degree is also bounded by $C(N, m)$:

$$G_{IK}(t, \xi, X) = \sum_{P, Q \quad |P_1| + |P_2| + |Q| \leq C(N, m+2)} A_{IK, P_1 P_2 Q} t^{P_1} \xi^{P_2} X^Q, \quad G_{IK}(t, \xi, f_{IK}(t, \xi)) \equiv 0$$

Write the solutions of $G_{IK} = 0$ as $f_{IK}^1 = f_{IK}, f_{IK}^2, \dots, f_{IK}^{n_{IK}}$, and define

$$\mathcal{F}(s, t, \xi, X) = \prod_{\substack{|I| + |K| \leq N \\ i_{IK} = 1}}^{n_{IK}} \sum_{|I| + |K| \leq N} f_{IK}^{i_{IK}}(t, \xi) s^I \xi^J X^K.$$

Then $\mathcal{F}(s, t, \xi, \bar{F}(s, t, \xi)) \equiv 0$. The coefficients of X in \mathcal{F} are symmetric polynomials of $\{f_{IK}^i\}_{i=1}^{n_{IK}}$ for each pair IK . It is thus a polynomial of $\frac{A_{IK, P_1 P_2 Q}}{A_{IK, 000}}$ for all IK and $P_1 P_2 Q$. While total degree of each f_{IK} is bounded by $C(N, m)$, the total degree of $\frac{A_{IK, PQ}}{A_{IK, 00}}$ is bounded by $C(N, m)$ by Lemma (A.3). Hence the total degree of the polynomial \mathcal{F} is bounded above by $C(N, m)$.

■

B Appendix 2: Affine cell coordinate functions for the two exceptional classes of the Hermitian symmetric spaces of compact type

Define the multiplication rule of octonions with the standard basis $\{e_0 = 1, e_1, \dots, e_7\}$ by the following table:

	e_1	e_2	e_4	e_7	e_3	e_6	e_5
e_1	-1	e_4	$-e_2$	$-e_3$	e_7	$-e_5$	e_6
e_2	$-e_4$	-1	e_1	$-e_6$	e_5	e_7	$-e_3$
e_4	e_2	$-e_1$	-1	$-e_5$	$-e_6$	e_3	e_7
e_7	e_3	e_6	e_5	-1	$-e_1$	$-e_2$	$-e_4$
e_3	$-e_7$	$-e_5$	e_6	e_1	-1	$-e_4$	e_2
e_6	e_5	$-e_7$	$-e_3$	e_2	e_4	-1	$-e_1$
e_5	$-e_6$	e_3	$-e_7$	e_4	$-e_2$	e_1	-1

Define

$$\begin{aligned}
x &= (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \\
y &= (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7) \\
A_0 &= A_0(x, y) = y_0x_0 + y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4 + y_5x_5 + y_6x_6 + y_7x_7 \\
A_1 &= A_1(x, y) = -y_0x_1 + y_1x_0 - y_2x_4 + y_4x_2 - y_3x_7 + y_7x_3 - y_5x_6 + y_6x_5 \\
A_2 &= A_2(x, y) = -y_0x_2 + y_2x_0 - y_4x_1 + y_1x_4 - y_3x_5 + y_5x_3 - y_6x_7 + y_7x_6 \\
A_3 &= A_3(x, y) = -y_0x_3 + y_3x_0 + y_1x_7 - y_7x_1 + y_2x_5 - y_5x_2 - y_4x_6 + y_6x_4 \\
A_4 &= A_4(x, y) = -y_0x_4 + y_4x_0 - y_1x_2 + y_2x_1 + y_3x_6 - y_6x_3 - y_5x_7 + y_7x_5 \\
A_5 &= A_5(x, y) = -y_0x_5 + y_5x_0 + y_1x_6 - y_6x_1 - y_2x_3 + y_3x_2 + y_4x_7 - y_7x_4 \\
A_6 &= A_6(x, y) = -y_0x_6 + y_6x_0 - y_1x_5 + y_5x_1 + y_2x_7 - y_7x_2 - y_3x_4 + y_4x_3 \\
A_7 &= A_7(x, y) = -y_0x_7 + y_7x_0 - y_1x_3 + y_3x_1 - y_2x_6 + y_6x_2 - y_4x_5 + y_5x_4 \\
B_0 &= B_0(x, y) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 \\
B_1 &= B_1(x, y) = y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2
\end{aligned}$$

The embedding functions of a Zariski open subset \mathcal{A} , which is identified with \mathbb{C}^{16} with coordinates $z := (x_0, \dots, x_7, y_0, \dots, y_7)$, of $M_{16} := \frac{E_6}{SO(10) \times SO(2)}$ into \mathbb{P}^{26} are given by:

$$z \rightarrow [1, x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, B_0, B_1]$$

Define

$$\begin{aligned}
x &= (x_1, x_2, x_3) \\
y &= (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7) \\
t &= (t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7) \\
\omega &= (\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) \\
A = A(x, y, t, \omega) &= x_2 x_3 - (\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2) \\
B = B(x, y, t, \omega) &= x_1 x_3 - (t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2) \\
C = C(x, y, t, \omega) &= x_1 x_2 - (y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2) \\
D_0 = D_0(x, y, t, \omega) &= t_0 \omega_0 + t_1 \omega_1 + t_2 \omega_2 + t_3 \omega_3 + t_4 \omega_4 + t_5 \omega_5 + t_6 \omega_6 + t_7 \omega_7 - x_3 y_0 \\
D_1 = D_1(x, y, t, \omega) &= - t_0 \omega_1 + t_1 \omega_0 - t_2 \omega_4 + t_4 \omega_2 - t_3 \omega_7 + t_7 \omega_3 - t_5 \omega_6 + t_6 \omega_5 - x_3 y_1 \\
D_2 = D_2(x, y, t, \omega) &= - t_0 \omega_2 + t_2 \omega_0 - t_4 \omega_1 + t_1 \omega_4 - t_3 \omega_5 + t_5 \omega_3 - t_6 \omega_7 + t_7 \omega_6 - x_3 y_2 \\
D_3 = D_3(x, y, t, \omega) &= - t_0 \omega_3 + t_3 \omega_0 + t_1 \omega_7 - t_7 \omega_1 + t_2 \omega_5 - t_5 \omega_2 - t_4 \omega_6 + t_6 \omega_4 - x_3 y_3 \\
D_4 = D_4(x, y, t, \omega) &= - t_0 \omega_4 + t_4 \omega_0 - t_1 \omega_2 + t_2 \omega_1 + t_3 \omega_6 - t_6 \omega_3 - t_5 \omega_7 + t_7 \omega_5 - x_3 y_4 \\
D_5 = D_5(x, y, t, \omega) &= - t_0 \omega_5 + t_5 \omega_0 + t_1 \omega_6 - t_6 \omega_1 - t_2 \omega_3 + t_3 \omega_2 + t_4 \omega_7 - t_7 \omega_4 - x_3 y_5 \\
D_6 = D_6(x, y, t, \omega) &= - t_0 \omega_6 + t_6 \omega_0 - t_1 \omega_5 + t_5 \omega_1 + t_2 \omega_7 - t_7 \omega_2 - t_3 \omega_4 + t_4 \omega_3 - x_3 y_6 \\
D_7 = D_7(x, y, t, \omega) &= - t_0 \omega_7 + t_7 \omega_0 - t_1 \omega_3 + t_3 \omega_1 - t_2 \omega_6 + t_6 \omega_2 - t_4 \omega_5 + t_5 \omega_4 - x_3 y_7 \\
E_0 = E_0(x, y, t, \omega) &= y_0 \omega_0 - y_1 \omega_1 - y_2 \omega_2 - y_3 \omega_3 - y_4 \omega_4 - y_5 \omega_5 - y_6 \omega_6 - y_7 \omega_7 - x_2 t_0 \\
E_1 = E_1(x, y, t, \omega) &= y_0 \omega_1 + y_1 \omega_0 + y_2 \omega_4 - y_4 \omega_2 + y_3 \omega_7 - y_7 \omega_3 + y_5 \omega_6 - y_6 \omega_5 - x_2 t_1 \\
E_2 = E_2(x, y, t, \omega) &= y_0 \omega_2 + y_2 \omega_0 + y_4 \omega_1 - y_1 \omega_4 + y_3 \omega_5 - y_5 \omega_3 + y_6 \omega_7 - y_7 \omega_6 - x_2 t_2 \\
E_3 = E_3(x, y, t, \omega) &= y_0 \omega_3 + y_3 \omega_0 - y_1 \omega_7 + y_7 \omega_1 - y_2 \omega_5 + y_5 \omega_2 + y_4 \omega_6 - y_6 \omega_4 - x_2 t_3 \\
E_4 = E_4(x, y, t, \omega) &= y_0 \omega_4 + y_4 \omega_0 + y_1 \omega_2 - y_2 \omega_1 - y_3 \omega_6 + y_6 \omega_3 + y_5 \omega_7 - y_7 \omega_5 - x_2 t_4 \\
E_5 = E_5(x, y, t, \omega) &= y_0 \omega_5 + y_5 \omega_0 - y_1 \omega_6 + y_6 \omega_1 + y_2 \omega_3 - y_3 \omega_2 - y_4 \omega_7 + y_7 \omega_4 - x_2 t_5 \\
E_6 = E_6(x, y, t, \omega) &= y_0 \omega_6 + y_6 \omega_0 + y_1 \omega_5 - y_5 \omega_1 - y_2 \omega_7 + y_7 \omega_2 + y_3 \omega_4 - y_4 \omega_3 - x_2 t_6 \\
E_7 = E_7(x, y, t, \omega) &= y_0 \omega_7 + y_7 \omega_0 + y_1 \omega_3 - y_3 \omega_1 + y_2 \omega_6 - y_6 \omega_2 + y_4 \omega_5 - y_5 \omega_4 - x_2 t_7 \\
F_0 = F_0(x, y, t, \omega) &= y_0 t_0 + y_1 t_1 + y_2 t_2 + y_3 t_3 + y_4 t_4 + y_5 t_5 + y_6 t_6 + y_7 t_7 - x_1 \omega_0 \\
F_1 = F_1(x, y, t, \omega) &= y_0 t_1 - y_1 t_0 - y_2 t_4 + y_4 t_2 - y_3 t_7 + y_7 t_3 - y_5 t_6 + y_6 t_5 - x_1 \omega_1 \\
F_2 = F_2(x, y, t, \omega) &= y_0 t_2 - y_2 t_0 - y_4 t_1 + y_1 t_4 - y_3 t_5 + y_5 t_3 - y_6 t_7 + y_7 t_6 - x_1 \omega_2 \\
F_3 = F_3(x, y, t, \omega) &= y_0 t_3 - y_3 t_0 + y_1 t_7 - y_7 t_1 + y_2 t_5 - y_5 t_2 - y_4 t_6 + y_6 t_4 - x_1 \omega_3 \\
F_4 = F_4(x, y, t, \omega) &= y_0 t_4 - y_4 t_0 - y_1 t_2 + y_2 t_1 + y_3 t_6 - y_6 t_3 - y_5 t_7 + y_7 t_5 - x_1 \omega_4 \\
F_5 = F_5(x, y, t, \omega) &= y_0 t_5 - y_5 t_0 + y_1 t_6 - y_6 t_1 - y_2 t_3 + y_3 t_2 + y_4 t_7 - y_7 t_4 - x_1 \omega_5 \\
F_6 = F_6(x, y, t, \omega) &= y_0 t_6 - y_6 t_0 - y_1 t_5 + y_5 t_1 + y_2 t_7 - y_7 t_2 - y_3 t_4 + y_4 t_3 - x_1 \omega_6 \\
F_7 = F_7(x, y, t, \omega) &= y_0 t_7 - y_7 t_0 - y_1 t_3 + y_3 t_1 - y_2 t_6 + y_6 t_2 - y_4 t_5 + y_5 t_4 - x_1 \omega_7
\end{aligned}$$

$$\begin{aligned}
G = G(x, y, t, \omega) = & x_1 x_2 x_3 - x_1(\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2) \\
& - x_2(t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2) \\
& - x_3(y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2) \\
& + 2\{(y_0\omega_0 - y_1\omega_1 - y_2\omega_2 - y_3\omega_3 - y_4\omega_4 - y_5\omega_5 - y_6\omega_6 - y_7\omega_7)t_0 \\
& + (y_0\omega_1 + y_1\omega_0 + y_2\omega_4 - y_4\omega_2 + y_3\omega_7 - y_7\omega_3 + y_5\omega_6 - y_6\omega_5)t_1 \\
& + (y_0\omega_2 + y_2\omega_0 + y_4\omega_1 - y_1\omega_4 + y_3\omega_5 - y_5\omega_3 + y_6\omega_7 - y_7\omega_6)t_2 \\
& + (y_0\omega_3 + y_3\omega_0 - y_1\omega_7 + y_7\omega_1 - y_2\omega_5 + y_5\omega_2 + y_4\omega_6 - y_6\omega_4)t_3 \\
& + (y_0\omega_4 + y_4\omega_0 + y_1\omega_2 - y_2\omega_1 - y_3\omega_6 + y_6\omega_3 + y_5\omega_7 - y_7\omega_5)t_4 \\
& + (y_0\omega_5 + y_5\omega_0 - y_1\omega_6 + y_6\omega_1 + y_2\omega_3 - y_3\omega_2 - y_4\omega_7 + y_7\omega_4)t_5 \\
& + (y_0\omega_6 + y_6\omega_0 + y_1\omega_5 - y_5\omega_1 - y_2\omega_7 + y_7\omega_2 + y_3\omega_4 - y_4\omega_3)t_6 \\
& + (y_0\omega_7 + y_7\omega_0 + y_1\omega_3 - y_3\omega_1 + y_2\omega_6 - y_6\omega_2 + y_4\omega_5 - y_5\omega_4)t_7\}
\end{aligned}$$

The embedding functions of a Zariski open subset \mathcal{A} , which is identified with \mathbb{C}^{27} with coordinates $z := (x, y, t, \omega) = (x_1, x_2, x_3, y_0, \dots, y_7, t_0, \dots, t_7, \omega_0, \dots, \omega_7)$, of $M_{27} := \frac{E_7}{E_6 \times SO(2)}$ into \mathbb{P}^{55} are given by: $z \rightarrow [1, x, y, t, \omega, A, B, C, D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7, E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7, F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, G]$. The detailed discussions related to this Appendix can be found in [CMP] and [Fr].

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